Question 1: *Fourier Transforming a Differential Equation*

As shown in the lecture, one of the crucial equations to solve when looking at the emission of radiation from accelerated charges has the form

$$\nabla^2\Psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{x},t)}{\partial t^2} = -4\pi f(\mathbf{x},t) \tag{4.22}$$

which can be solved using a Green's functions Ansatz, where the Green's function $G(\mathbf{x},t;\mathbf{x}',t')$ is the solution of

$$\nabla^2 G(\mathbf{x},t;\mathbf{x}',t') - \frac{1}{c^2}\frac{\partial^2 G(\mathbf{x},t;\mathbf{x}',t')}{\partial t^2} = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{4.23}$$

The solutions of this equation can be best obtained when working in Fourier space, where the Fourier transform is defined through

$$f(\mathbf{x},\omega) = \int f(\mathbf{x},t)\mathrm{e}^{+i\omega t}\mathrm{d}t \quad \text{and} \quad G(\mathbf{x},\omega;\mathbf{x}',t') = \int G(\mathbf{x},t;\mathbf{x}',t')\mathrm{e}^{+i\omega t}\mathrm{d}t \tag{4.25}$$

and where the inverse transform is

$$f(\mathbf{x},t) = \frac{1}{2\pi}\int f(\mathbf{x},\omega)\mathrm{e}^{-i\omega t}\mathrm{d}\omega \quad \text{and} \quad G(\mathbf{x},t;\mathbf{x}',t') = \frac{1}{2\pi}\int G(\mathbf{x},\omega;\mathbf{x}',t')\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{4.27}$$

a) By inserting the inverse Fourier transform of $G(\mathbf{x},t;\mathbf{x}',t')$, show that

$$\nabla^2 G(\mathbf{x},t;\mathbf{x}',t') = \frac{1}{2\pi}\int \left(\nabla^2 G(\mathbf{x},\omega;\mathbf{x}',t')\right)\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{4.30}$$

b) By inserting the inverse Fourier transform of $G(\mathbf{x},t;\mathbf{x}',t')$, show that

$$-\frac{1}{c^2}\frac{\partial^2 G(\mathbf{x},t;\mathbf{x}',t')}{\partial t^2} = \frac{1}{2\pi}\int k^2 G(\mathbf{x},\omega;\mathbf{x}',t')\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{4.33}$$

where $k = \omega/c$.

c) Making use of one of the definitions of the δ-function,

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{i\omega t}\mathrm{d}\omega \tag{4.34}$$

(which you can only use if you are not a mathematician…) and of

$$\delta(t-t') = \delta(t'-t) \tag{s1.1}$$

show that

$$(-4\pi)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') = \frac{1}{2\pi}\int(-4\pi)\delta(\mathbf{x}-\mathbf{x}')\mathrm{e}^{i\omega t'}\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{4.36}$$