Friedrich–Alexander–Universität Erlangen–Nürnberg

Astrophysical Radiation Processes
Sommersemester 2008
Worksheet 2

Question 1: *Properties of the Planck Spectrum*

a) Planck's formula in *frequency space* was shown in the lectures to be

$$\frac{\mathrm{d}E}{\mathrm{d}A\mathrm{d}t\mathrm{d}\Omega\mathrm{d}\nu} = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1} \tag{3.18}$$

Note that

$$\frac{\mathrm{d}E}{\mathrm{d}A\mathrm{d}t\mathrm{d}\Omega} = B_\nu \mathrm{d}\nu \tag{w1.1}$$

which is independent of frequency. It is easy to see that a similar relationship has to hold for B_λ, such that

$$|B_\nu \mathrm{d}\nu| = |B_\lambda \mathrm{d}\lambda| \tag{w1.2}$$

Using the previous equation, show that in *wavelength space*

$$B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT)-1} \tag{w1.3}$$

b) *Derivation of the Rayleigh-Jeans-Law:* Using

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + \ldots \tag{3.48}$$

show that

$$B_\nu \approx \frac{2\nu^2}{c^2}kT \tag{3.49}$$

c) *Derivation of the Wien spectrum*: Using

$$\exp\left(\frac{h\nu}{kT}\right) - 1 \sim \exp\left(\frac{h\nu}{kT}\right) \tag{3.51}$$

show that for $h\nu \gg kT$

$$B_\nu \sim \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right) \tag{3.52}$$

d) The energy density for radiation was found to be

$$u_\nu(\Omega) = \frac{1}{c}I_\nu \tag{2.71}$$

i. Show that for isotropic radiation

$$u_\nu = \int_{4\pi\,\mathrm{sr}} u\,\mathrm{d}\Omega = \frac{4\pi}{c}I_\nu \tag{s1.1}$$

ii. Based on the above, show that the total energy density of black body radiation is

$$u_{\mathrm{BB}}(T) = \int_0^\infty \frac{4\pi}{c}B_\nu\,\mathrm{d}\nu = \frac{8\pi^5}{15}\left(\frac{kT}{hc}\right)^3 kT = aT^4 \tag{3.57}$$

Hint: Substitute $x = h\nu/kT$ and note that $\int_0^\infty \frac{x^3}{\exp(x)-1} = \pi^4/15$.