## Emitted spectrum

So far we have calculated $E(t)$, but what we really want to know is the emitted spectrum, i.e., the amount of energy radiated between $\omega$ and $\omega + \,\mathrm{d}\omega$.

We have just found for the energy emitted into $\mathrm{d}\Omega$:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{\ddot{\boldsymbol{d}}^2}{4\pi c^3}\sin^2\theta \tag{4.92}$$

Because of Parseval's theorem (see worksheet 4), the total emitted energy is

$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{\sin^3\theta}{4\pi c^3}\int_{-\infty}^{+\infty}|\ddot{d}(t)|^2\,\mathrm{d}t = \frac{\sin^3\theta}{8\pi^2 c^3}\int_{-\infty}^{+\infty}|\ddot{d}(\omega)|^2\,\mathrm{d}\omega = \frac{\sin^3\theta}{4\pi^2 c^3}\int_{0}^{+\infty}|\ddot{d}(\omega)|^2\,\mathrm{d}\omega \tag{4.93}$$

but $\ddot{d}(\omega) = \omega^2 d(\omega)$ (see worksheet 5), such that

$$\frac{\mathrm{d}E}{\mathrm{d}\Omega} = \frac{\sin^3\theta}{4\pi^2 c^3}\int_{0}^{+\infty}\omega^4|d(\omega)|^2\,\mathrm{d}\omega \tag{4.94}$$

and therefore

$$\frac{\mathrm{d}E}{\mathrm{d}\omega\,\mathrm{d}\Omega} = \frac{\omega^4|d(\omega)|^2}{4\pi^2 c^3}\sin^3\theta \tag{4.95}$$

integrating over $\mathrm{d}\Omega$ and remembering $\langle\sin^2\theta\rangle = 2/3$ gives the total spectral energy density

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \frac{2}{3\pi}\frac{\omega^4}{c^3}|d(\omega)|^2 \tag{4.96}$$





# *Bremsstrahlung*



## Introduction

*Bremsstrahlung:* Radiation emitted from particle accelerated by Coulomb field of another charge

Alternative name: free-free radiation

Detailed understanding will require quantum electrodynamics, but we can get a good overview using classical electrodynamics. . .

*Classical and QM derivations:*

Oster, 1961, Rev. Mod. Phys. 33(4), 525

Scheuer, 1960, MNRAS 129, 231

*QED derivations:*

electron-proton bremsstrahlung:

Karzas& Latter, 1961, ApJ Suppl. 6, 167

Blumenthal& Gould, 1970, Rev. Mod. Phys., 42(2), 237

electron-electron bremsstrahlung:

Haug, 1989, A&A 218, 330

electron-positron bremsstrahlung:

Haug, 1987, A&A 178, 292





## Introduction

General plan of attack:

1. use classical E&M (Larmor in the dipole approximation) to obtain overall behavior
2. Take quantum mechanics into account by introducing a Gaunt factor into the classical formulae

Dipole approximation o.k. for electron-nucleus bremsstrahlung.

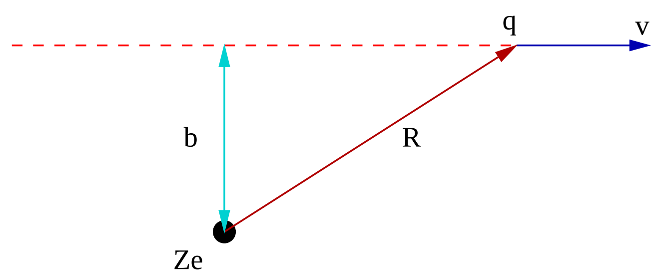
For electron-electron, proton-proton, etc., dipole moment is proportional to center of mass $\Longrightarrow \ddot{d} = 0 \Longrightarrow$ Calculation is (much) more difficult $\Longrightarrow$ See references.

But usually electron-proton bremsstrahlung is the dominant one.

When looking at electron-nucleus interaction, acceleration on electron is the most significant one

$\Longrightarrow$ Bremsstrahlung is mainly produced by electrons

## Single Speed Electrons, I




Work in classical analogue of Born approximation, the "small angle approximation": Path of electron not influenced by nucleus $\Longrightarrow$ Calculate motion along a straight line, $z = vt$.

Acceleration is mainly $\perp$ motion, i.e.,

$$|\ddot{\boldsymbol{x}}| = \frac{Z_i e^2}{m_\mathrm{e}(b^2 + v^2 t^2)} \tag{5.1}$$

($b$: impact parameter). Therefore, using Eq. (4.5), the $E$-field and its frequency dependence are

$$E(t) = \frac{Z_i e^3 \sin\theta}{m_\mathrm{e} c^2 R (b^2 + v^2 t^2)} \xrightarrow{\text{Fourier transform}} E(\omega) = \left(\frac{Z_i e^3 \sin\theta}{m_\mathrm{e} c^2 R}\right)\left(\frac{\pi}{bv}\right) \mathrm{e}^{-\omega b/v} \tag{5.2}$$





## Single Speed Electrons, II

The emitted energy per unit area and frequency is

$$\frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}\omega} = c|E(\omega)|^2 \tag{5.3}$$

The derivation of this equation is similar to Eq. (4.95)

Therefore the emitted energy is

$$\frac{\mathrm{d}W(b)}{\mathrm{d}\omega} = \int_0^\pi c|E(\omega)|^2 R^2 2\pi \sin\theta\,\mathrm{d}\theta = \frac{8}{3\pi}\left(\frac{Z_i^2 e^6}{m_\mathrm{e}^2 c^3}\right)\frac{1}{(bv)^2}\mathrm{e}^{-2\omega b/v} \tag{5.4}$$

Approximating the e-function with a step function,

$$\frac{\mathrm{d}W(b)}{\mathrm{d}\omega} = \begin{cases} \frac{8}{3\pi}\left(\frac{Z_i^2 e^6}{m_\mathrm{e}^2 c^3}\right)\frac{1}{(bv)^2} & \text{for } b \ll v/\omega \\ 0 & \text{for } b \gg v/\omega \end{cases} \tag{5.5}$$





## Single Speed Electrons, III

Assume electron density, $n_\mathrm{e}$, ion density, $n_\mathrm{i}$.

- Flux of electrons with velocity $v$ incident on ion: $n_\mathrm{e} v$.
- area element around ion: $2\pi b\,\mathrm{d}b$.

$\Longrightarrow$ emission per unit time, volume and frequency:

$$\frac{\mathrm{d}W}{\mathrm{d}\nu\,\mathrm{d}V\,\mathrm{d}t} = n_\mathrm{e} n_\mathrm{i} 2\pi v \int_{b_\mathrm{min}}^\infty \frac{\mathrm{d}W(b)}{\mathrm{d}\nu} b\,\mathrm{d}b \tag{5.6}$$

where $b_\mathrm{min}$ is some minimum impact parameter.

Most colissions will be small angle collissions

$\Longrightarrow$ Large contribution only up to a $b = b_\mathrm{max}$

$\Longrightarrow$ can use small angle approximation, Eq. (5.5):

$$\frac{\mathrm{d}W}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} = \frac{16 e^6}{3c^3 m_\mathrm{e}^2 v} n_\mathrm{e} n_\mathrm{i} Z^2 \int_{b_\mathrm{min}}^{b_\mathrm{max}} \frac{\mathrm{d}b}{b} = \frac{16 e^6}{3c^3 m_\mathrm{e}^2 v} n_\mathrm{e} n_\mathrm{i} Z^2 \ln\left(\frac{b_\mathrm{max}}{b_\mathrm{min}}\right) \tag{5.7}$$





## Single Speed Electrons, IV

What are values of $b_\mathrm{min}$ and $b_\mathrm{max}$?

Since they occur in a logarithm, precise values are not necessary.

For $b_\mathrm{max}$, ensure that small angle approximation holds, i.e.,

$$b_\mathrm{max} = \frac{v}{\omega} \tag{5.8}$$

For $b_\mathrm{min}$, we have to consider two approaches:

1. straight-line approximation should hold, i.e., want small $\Delta v$. Define "small" by $\Delta v \sim v$, which holds when

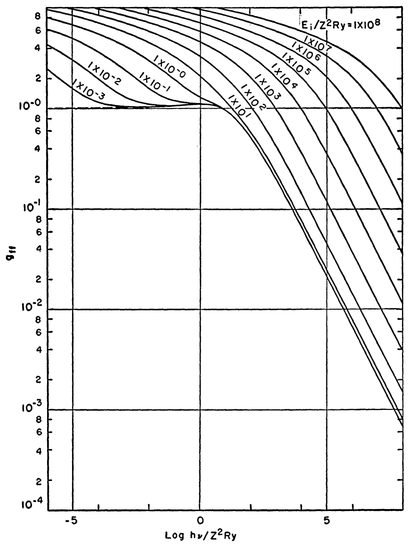
$$b_\mathrm{min}^1 = \frac{4Ze^2}{\pi m v^2} \tag{5.9}$$

2. particle approximation should hold, i.e., quantum effects better be small $\Longrightarrow$ uncertainty principle, $\Delta x \Delta p \gtrsim \hbar$, with $\Delta x \sim b$, $\Delta p \sim m_\mathrm{e} v$,

$$b_\mathrm{min}^2 = \frac{h}{mv} \tag{5.10}$$

and use larger of both.

## Single Speed Electrons, V




In reality, perform QED computations. Result is

$$\frac{dW}{d\omega\, dV\, dt} = \frac{16\pi e^6}{3^{3/2} m_e^2 c^3 v} n_e n_i Z^2 g_{ff}(v,\omega) \quad (5.11)$$

where $g_{ff}$ is the Gaunt factor.

From the above, one can formally write

$$g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{max}}{b_{min}}\right) \quad (5.12)$$

Karzas&Latter, 1961, Fig. 1





## Thermal Bremsstrahlung

Emission from a plasma of thermal particles: thermal bremsstrahlung.

Assume Maxwell-Boltzmann velocity distribution, i.e., probability for particle being in speed range $v \dots v + dv$:

$$dP \propto v^2 \exp\left(-\frac{mv^2}{2kT}\right) \quad (5.13)$$

To obtain total Bremsstrahlung emissivity, integrate Eq. (5.11) over electron velocity distribution.

Note that in order to emit photon of energy $h\nu$, we need an electron kinetic energy

$$\frac{1}{2} m_e v^2 \gg h\nu \quad \Longrightarrow \quad v_{min} = \sqrt{\frac{2h\nu}{m_e}} \quad (5.14)$$

(photon discreteness effect).

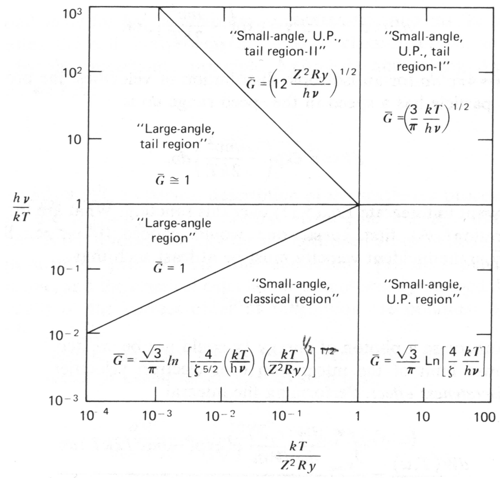
Thus

$$\frac{dW(T,\omega)}{dV\, dt\, d\omega} = \frac{1}{\int_0^\infty v^2 \exp(-mv^2/2kT)\, dv} \int_{v_{min}}^\infty \frac{dW(v,\omega)}{d\omega\, dV\, dt} v^2 \exp(-mv^2/2kT)\, dv \quad (5.15)$$





## Thermal Bremsstrahlung




Rybicki&Lightman, Fig. 5.2 (note correction of typo);
1 Ry=13.6 eV, $\bar G = \bar g_{ff}$

Substitute $d\omega = 2\pi\, d\nu$ into Eq. (5.15) to obtain

$$\frac{dW}{dV\, dt\, d\nu} = \frac{2^5 \pi e^6}{3 m_e c^3} \left(\frac{2\pi}{3 m_e k}\right)^{1/2} \cdot T^{-1/2} Z^2 n_e n_i\, e^{-h\nu/kT} \cdot \bar g_{ff} \quad (5.16)$$

where $\bar g_{ff}$ is the velocity averaged Gaunt factor.

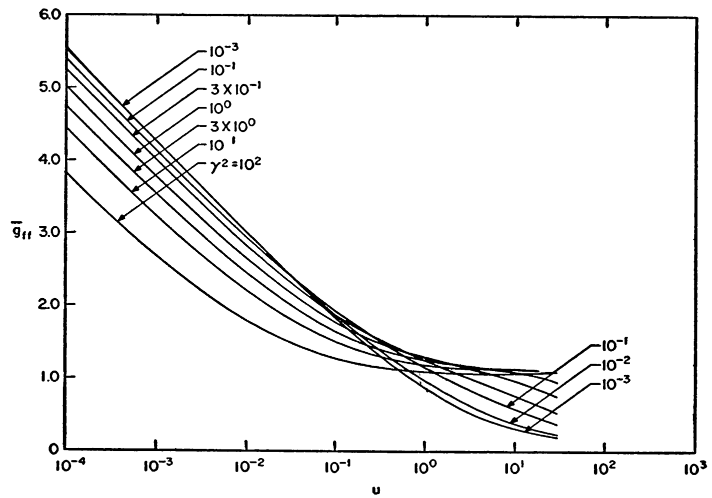
In real numbers, the emissivity is

$$j_\nu^{ff} = 6.8 \times 10^{-38}\,\mathrm{erg\,s^{-1}\,cm^{-3}} \cdot Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar g_{ff} \quad (5.17)$$



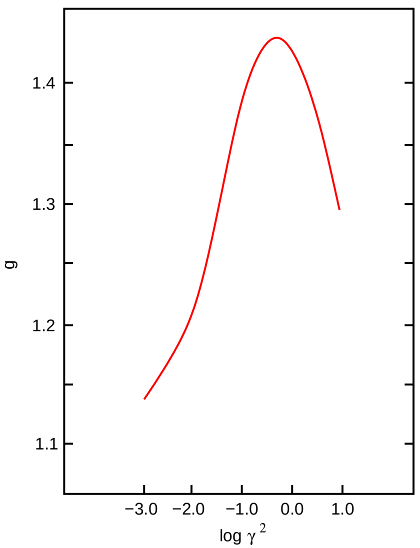


## Thermal Bremsstrahlung




Gaunt factor, $\bar g_{ff}$, from Karzas & Latter, Fig. 5. $u = h\nu/kT$, for various values of $\gamma^2 = Z^2\,\mathrm{Ry}/kT$, where 1 Ry = 13.6 eV.

## Thermal Bremsstrahlung




The total power emitted per volume is found by integrating spectrum over frequency:

$$j^{\mathrm{ff}} = \frac{\mathrm{d}W}{\mathrm{d}t\,\mathrm{d}V} = \left(\frac{2\pi kT}{3m_\mathrm{e}}\right)^{1/2} \frac{2^5\pi e^6}{3hm_\mathrm{e}c^3} Z^2 n_\mathrm{e} n_\mathrm{i} \bar{g}_\mathrm{B}(T) \qquad (5.18)$$

where $\bar{g}_\mathrm{B}(T)$ is the frequency average of $g_\mathrm{ff}$.
To a precision of $\sim$ 20%, $\bar{g}_\mathrm{B}(T) = 1.2$.

$\bar{g}_\mathrm{B}$ as a function of $\gamma = Z^2\,\mathrm{Ry}/kT$ (after Karzas & Latter, Fig. 6)





## Kirchhoff's law

Inverse process to free-free emission: thermal bremsstrahlung absorption, i.e., absorption of energy by free moving electron. Because of a nice trick, this complicated process can be very easily described with what we know so far.

Assume thermal equilibrium. Then, by definition the source function must be a black body:

$$S_\nu = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1} \qquad (2.99)$$

But the source function was also

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \qquad (2.92)$$

Such that

$$j_\nu = \alpha_\nu B_\nu \qquad (5.19)$$

This is Kirchhoff's law.

Kirchhoff's law is a valid relationship for *all* physical conditions, since $\alpha_\nu$, $j_\nu$ do not depend on the fact that the derivation was based on thermodynamic equilibrium.





## Bremsstrahlung Absorption

For bremsstrahlung, $j_\nu^{\mathrm{ff}}$ is given by

$$j_\nu^{\mathrm{ff}} = \frac{1}{4\pi}\frac{\mathrm{d}W}{\mathrm{d}t\,\mathrm{d}V\,\mathrm{d}\nu} \qquad (5.20)$$

and we can use Eqs. 5.16 and 5.19 to determine $\alpha_\nu$:

$$\alpha_\nu^{\mathrm{ff}} = \frac{4e^6}{3m_\mathrm{e}hc}\left(\frac{2\pi}{3km_\mathrm{e}}\right)^{1/2} T^{-1/2} Z^2 n_\mathrm{e} n_\mathrm{i} \nu^{-3}\left(1 - \mathrm{e}^{-h\nu/kT}\right)\bar{g}_\mathrm{ff} \qquad (5.21)$$

Note especially two asymptotical relations:

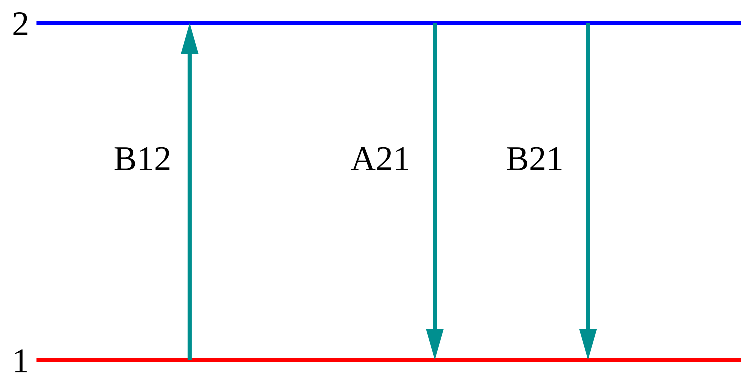
$$\alpha_\nu \propto \begin{cases} n_\mathrm{i} n_\mathrm{e} g T^{-1/2}\nu^{-3} & \text{for } h\nu \gg kT \\ n_\mathrm{i} n_\mathrm{e} g T^{-3/2}\nu^{-2} & \text{for } h\nu \ll kT \end{cases} \qquad (5.22)$$

The interpretation of the exponential factor is deeply related to microphysics.





## Principle of detailed balance




Describe microphysics with Einstein coefficients:

$A_{21}$: transition probability per unit time for spontaneous emission.
$B_{12}I_\nu$: transition probability for absorption per unit time.
$B_{21}I_\nu$: transition probability for stimulated emission per unit time.

## Principle of detailed balance

Assume state populations $n_1$, $n_2$. In thermodynamic equilibrium, spontaneous and induced emissions balance absorptions:

$$n_2 A_{21} + n_2 B_{21} I_\nu = n_1 B_{12} I_\nu \tag{5.23}$$

such that

$$I_\nu = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1} \tag{5.24}$$

But, in thermodynamic equilibrium, Boltzmann holds:

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(\frac{h\nu_{21}}{kT}\right) \tag{5.25}$$

where $g_1$, $g_2$: statistical weights. Therefore

$$I_\nu = \frac{A_{21}/B_{21}}{(g_1/g_2)(B_{12}/B_{21})\exp(h\nu_{21}/kT) - 1} = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1} \tag{5.26}$$

since, in thermodynamical equilibrium, $I_\nu = B_\nu$ by definition.
Therefore we obtain the Einstein relations (1916):

$$g_1 B_{12} = g_2 B_{21} \tag{5.27a}$$

$$A_{21} = \frac{2h\nu_{21}^3}{c^2} B_{21} \tag{5.27b}$$





## Principle of detailed balance

Now write radiation transfer equation using Einstein coefficients:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{h\nu_{21} n_2 A_{21}}{4\pi} - n_1 B_{12} h\nu_{21} I_\nu + n_2 B_{21} h\nu_{21} I_\nu$$
$$= \frac{h\nu_{21} n_2 A_{21}}{4\pi} - h\nu_{21}(n_1 B_{12} - n_2 B_{21}) I_\nu \tag{5.28}$$

As shown on worksheet 5, assuming thermodynamic equilibrium one finds

$$j_\nu = \frac{h\nu_{21} n_2 A_{21}}{4\pi} \quad \text{and} \quad \alpha_\nu = h\nu_{21} n_1 B_{12}\left(1 - \exp\left(-\frac{h\nu_{21}}{kT}\right)\right) \tag{5.29}$$

$\Longrightarrow$ The exponential term showing up in the bremsstrahlung self-absorption formula is due to stimulated emission!

$\Longrightarrow$ Absorption *with* $\exp(h\nu/kT)$-term (i.e., for $h\nu \gg kT$) is called absorption coefficient corrected for stimulated emission.

$\Longrightarrow$ Absorption *without* $\exp(h\nu/kT)$-term (i.e., for $h\nu \ll kT$) is called absorption coefficient *uncorrected* for stimulated emission.





## Self-absorption

An astronomical source radiating purely bremsstrahlung will show the effects of both, bremsstrahlung emission *and* absorption, in its spectrum.

Let's look at radio regime. Using Eq. (5.22) for the radio, $\alpha_\nu \propto T^{-3/2}\nu^{-2}$, such that the optical depth

$$\tau = \int \alpha_\nu \,\mathrm{d}s \propto T^{-3/2}\nu^{-2} \int n_\mathrm{e}^2 \,\mathrm{d}s \tag{5.30}$$

assuming $T =$ const. and where $n_\mathrm{e}$ is electron density. We assume pure hydrogen $\Longrightarrow n_\mathrm{p} = n_\mathrm{e}$.

A precise fit for frequencies $\lesssim$ 10 GHz is

$$\tau_\nu = 8.235 \times 10^{-2} T^{-1.35} \nu^{-2.1} \int n_\mathrm{e}^2 \,\mathrm{d}s \tag{5.31}$$

(Mezger & Henderson, 1967)

The integral

$$\mathrm{EM} = \int n_\mathrm{e}^2 \,\mathrm{d}s \tag{5.32}$$

is called the emission measure. It is typically expressed in units of $\mathrm{cm}^{-6}$ pc.





## Self-absorption

The equation of radiative transfer is

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu \tag{5.33}$$

Assume no back illumination.

Separation of variables:

$$\int_0^{I_\nu} \frac{\mathrm{d}I_\nu}{j_\nu - \alpha_\nu I_\nu} = \int_0^L \mathrm{d}s = L \tag{5.34}$$

where $L$ depth of cloud.

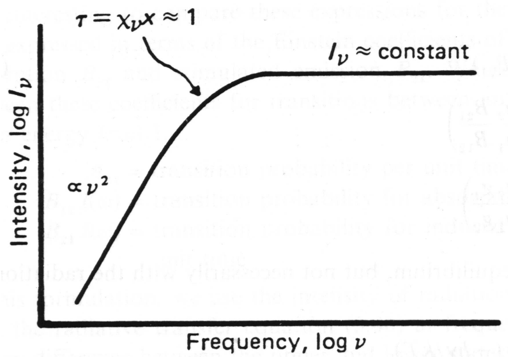
The integral has the value

$$L = \frac{1}{\alpha_\nu} \ln\left(\frac{j_\nu}{j_\nu - \alpha_\nu I_\nu}\right) \tag{5.35}$$

such that

$$I_\nu = \frac{j_\nu}{\alpha_\nu}\left(1 - \mathrm{e}^{-\alpha_\nu L}\right) = B_\nu\left(1 - \mathrm{e}^{-\tau_n u}\right) \tag{5.36}$$

## Self-absorption




For $\tau_\nu \lesssim 1$, i.e., $h\nu \gg kT$,

$$I_\nu \propto j_\nu \sim \text{const.} \tag{5.37}$$

For $\tau_\nu \gtrsim 1$, i.e., $h\nu \ll kT$,

$$I_\nu \sim \frac{j_\nu}{\alpha_\nu}\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right) \tag{5.38}$$

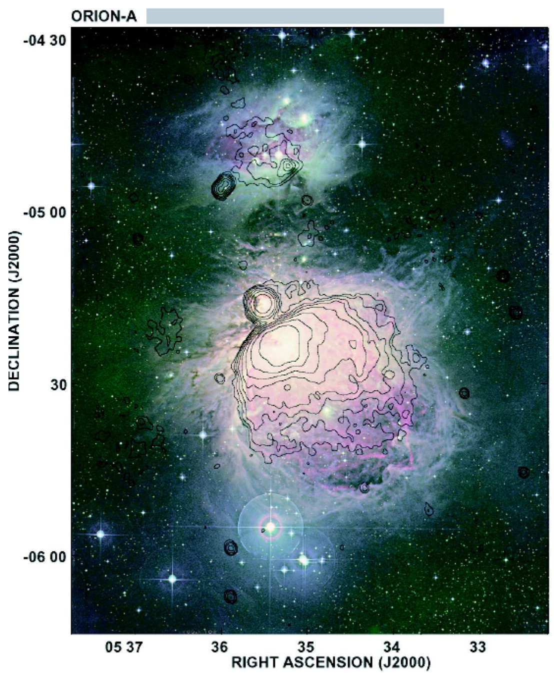
$$\sim \frac{2kT}{c^2}\nu^2 \tag{5.39}$$

Longair, Fig. 3.4

Thus, at high frequencies we would expect the spectrum of a bremsstrahlung emitting region to be $\sim$ flat, while for lower frequencies, spectrum $\propto \nu^2$ (Rayleigh Jeans!)

$\Longrightarrow$ Bremsstrahlung self-absorption.



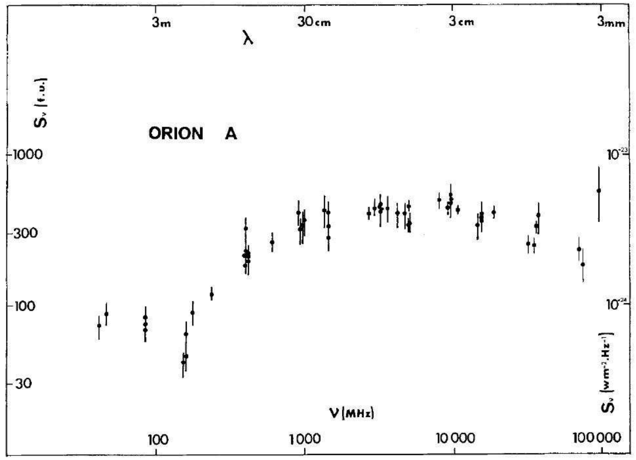



H II regions such as the Orion nebula have also strong emission in the radio band.

Subrahmanyan et al., (2001, Fig. 1)



## H II Regions, II




Goudis (1975, A&SS 36, 105)

Orion nebula has turnover at about 1 GHz,
$\Longrightarrow$ radio emission is dominated by bremsstrahlung.





## Cooling Flows

In centers of galaxy clusters: observe tenuous gas at high temperatures.

Assume hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Phi}{\mathrm{d}r} \quad \text{and} \quad P = \frac{\rho kT}{\mu m_\mathrm{H}} \tag{5.40}$$

Assume gravitational potential of the form

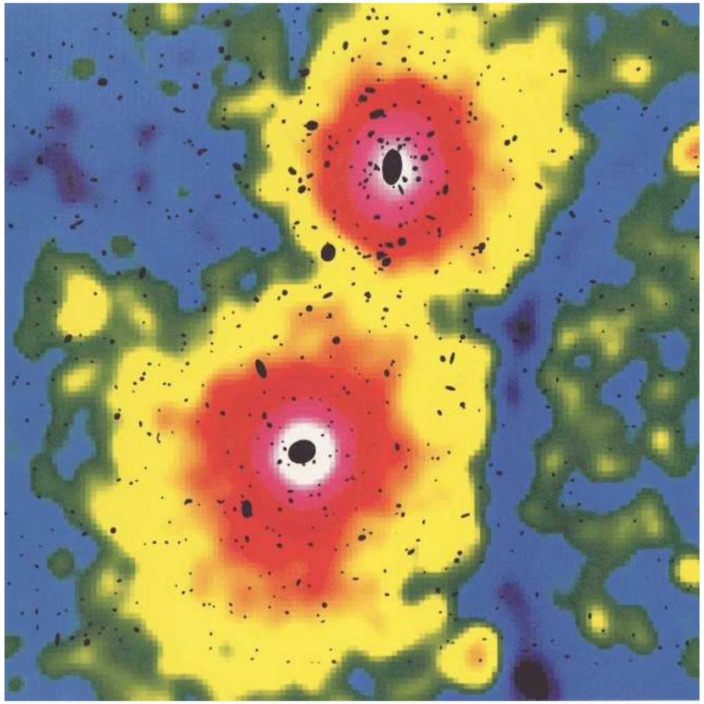
$$\Phi = \frac{9\sigma^2}{4\pi G r_\mathrm{c}^2}\left(1 + r/r_\mathrm{c}\right)^{-3/2} \tag{5.41}$$

(King profile).

Determine temperature $T(r)$ from observed X-ray spectrum.

In principle, this information is sufficient to reconstruct density profile of cluster.

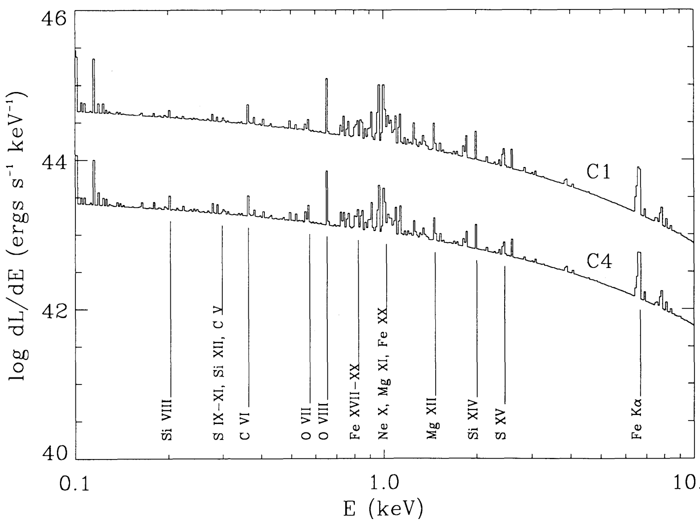
Cooling of gas $\propto j^\mathrm{ff} \propto \rho^2 \Longrightarrow$ Material cools $\Longrightarrow$ slides down gravitational potential $\Longrightarrow$ cooling flow.



Abell 3528 as observed with ROSAT (black: position/shape of optical galaxies.



## Cooling Flows



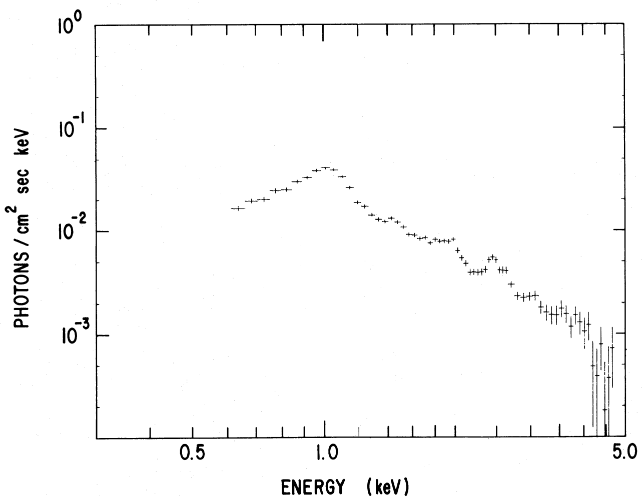

The theoretical X-ray spectra of cooling flows are dominated by bremsstrahlung and emission lines. . .

Wise & Sarazin, 1993





## Cooling Flows

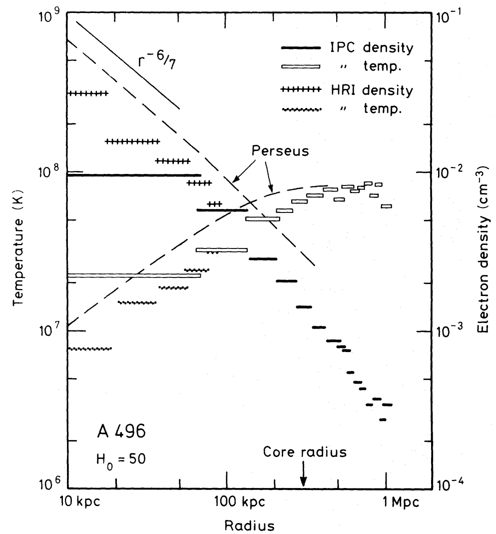



Lea et al., 1982 (Einstein SSS): galaxy clusters show bremsstrahlung continuum and lines.





## Cooling Flows




Temperature and Density inferred from bremsstrahlung fits to two cooling flow clusters: gas gets indeed denser and cooler towards the cluster center.

Note: recent observations with *Chandra* and *XMM-Newton* have revealed a large number of problems with the cooling flow interpretation. . .

Nulsen et al., 1982