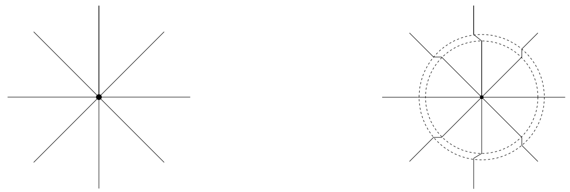
Radiation from Moving Charges

One of the most important results of classical electrodynamics is the fact that accelerated charges emit radiation. Here, we will derive the formula describing this result, "Larmor's formula", using a classical treatment due to J.J. Thomson and revived by Malcolm Longair. As shown after this, the *ab initio* derivation from Maxwell's equations gives the same result, but after a significantly more complicated calculation.

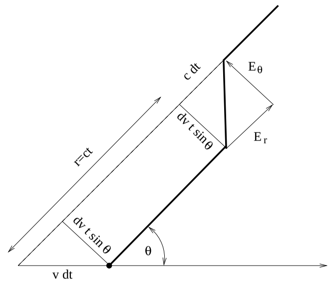
We start by taking a stationary charge at rest at time $t = 0$. The field lines from that charge have the simple configuration shown on the left side of the figure below.

We now accelerate the charge by a velocity difference Δv for a time interval Δt and look at the field line configuration at a time t after this.



Outside of a sphere of radius $r \sim ct$ the field lines still point towards the original position of the charge, because the information about the acceleration has not yet moved farther out than r. Inside of this sphere the field lines point towards the location the charge had after the acceleration. Properly spoken, the figure is not correct as the charge continued to drift with its new velocity since the end of the acceleration. The field lines in the figure should reflect this, but given that that information has not yet propagated out to r either, we can ignore it, and all of the computations we do below will only need information local ti r. Since the electric field lines have to be connected, there is a small region of width $c\Delta t$ in which the electric field has a non-radial component. An observer in this region will measure a temporal change in the E-field strength. By definition, a time dependent E-field is electromagnetic radiation and therefore this *Gedankenexperiment* has shown us that accelerating a charge will always lead to electromagnetic radiation!

To obtain the power radiated by the accelerated charge we need to know the strength of the electric field in the region of the electromagnetic pulse. This can be done by considering the following sketch of one E-field line:



where, for simplicity $\mathrm{d}t = \Delta t$ and $\mathrm{d}v = \Delta v$.

From simple geometry, one finds for the ratio of the electric fields in the pulse region:

$$\frac{E_\theta}{E_r} = \frac{\Delta v t \sin\theta}{c\Delta t} \tag{4.1}$$

E_r follows from Coulomb's law:

$$E_r = q \cdot \frac{1}{r^2} \tag{4.2}$$

and because we're observing at time t:

$$r = ct \tag{4.3}$$

Inserting E_r and r into Eq. (4.1) gives

$$E_\theta = E_r \frac{\Delta v}{\Delta t} \frac{t \sin\theta}{c} = q \frac{1}{r} \frac{1}{ct} \frac{\Delta v}{\Delta t} \frac{t \sin\theta}{c} \tag{4.4}$$

In the limit $\Delta t \to 0$, we can identify $\Delta v / \Delta t$ with the acceleration $\dot{v}$ (where the dot denotes differentiation with respect to time), and therefore we finally obtain

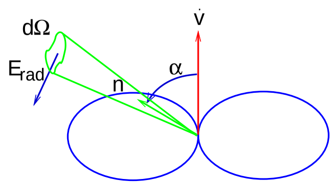
$$E_\theta = \frac{q}{rc^2} \dot{v} \sin\theta \tag{4.5}$$

So, during the pulse the E-field in the θ-direction changes from 0 to E_θ and back to 0. This is a pulse of electromagnetic radiation.

The energy flow per unit area and per second, S, is then given from Poynting's theorem, Eq. (2.32), as

$$S = \frac{c}{4\pi} E^2 = \frac{q^2}{4\pi c^3 r^2} \dot{v}^2 \sin^2\theta \tag{4.6}$$

This means that the energy loss has *dipolar form*, as shown in the following figure. Often, this equation is called *Larmor's formula* (although we will encounter another Larmor's formula below).



Note that the energy loss is symmetric, this means that the radiating particle is only loosing energy, but not momentum.

To obtain the total energy lost by the particle, we need to integrate the energy lost over all directions

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{4\pi\mathrm{sr}} S\, r^2\, \mathrm{d}\Omega = \int_{4\pi\mathrm{sr}} \frac{q^2}{4\pi c^3 r^2} \dot{v}^2 \sin^2\theta\, r^2\, \mathrm{d}\Omega \tag{4.7}$$

where $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ is the surface element of a sphere, θ and φ are the usual spherical coordinates, with θ going from 0 to π and φ from 0 to 2π. The total surface area of a sphere is 4π steradians. Performing the integral, lumping all constants into a helper variable A gives

$$\frac{\mathrm{d}E}{\mathrm{d}t} = A \int_{4\pi\mathrm{sr}} \sin^2\theta\, \mathrm{d}\Omega = A \int_0^\pi \sin^2\theta\, 2\pi \sin\theta\, \mathrm{d}\theta = 2\pi A \int_0^\pi \sin^3\theta\, \mathrm{d}\theta = \frac{8\pi}{3} A \tag{4.8}$$

And therefore we obtain *Larmor's formula* for the energy loss of an accelerated charge:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{8\pi}{3} \frac{q^2}{4\pi c^3} \dot{v}^2 = \frac{2q^2}{3c^3} \dot{v}^2 \tag{4.9}$$

Reminder: Maxwell's Equations

Reminder: In classical electrodynamics, electromagnetism is described by Maxwell's equations:

Coulomb's law:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \tag{2.1}$$

Law of Induction (Faraday):

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.2}$$

Nonexistence of Magnetic Monopoles (Gilbert):

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.3}$$

Ampère's law:

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \tag{2.4}$$

Lorentz Gauge, I

Because $\nabla \cdot \boldsymbol{B} = 0$, we can express $\boldsymbol{B}$ in terms of the vector potential, $\boldsymbol{A}$:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{4.10}$$

Therefore Faraday's law is

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \boldsymbol{A}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right) \tag{4.11}$$

Without changing $\boldsymbol{E}$, we can add an arbitrary function $-\nabla\phi$:

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \tag{4.12}$$

Both, $\boldsymbol{A}$ and ϕ are arbitrary up to the gradient of any scalar function Ψ.
$\Longrightarrow$ Allows the gauge transform

$$\boldsymbol{A}' \longrightarrow \boldsymbol{A} + \nabla\Psi \quad \text{and} \quad \phi' \longrightarrow \phi - \frac{1}{c}\frac{\partial \Psi}{\partial t} \tag{4.13}$$

where $\Psi(\boldsymbol{x}, t)$ completely arbitrary.

Here, will choose Ψ such that the Lorentz gauge holds:

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \tag{4.14}$$

Lorentz Gauge, II

Substitute Eqn. (4.10) into Coulomb's law to get an equation for ϕ:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \quad \longrightarrow \quad -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = 4\pi\rho \tag{4.15}$$

which is, because of the Lorentz Gauge Eq. (4.14):

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho \tag{4.16}$$

Ampère's Law gives an equation for $\boldsymbol{A}$:

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \quad \longrightarrow \quad \nabla \times (\nabla \times \boldsymbol{A}) = \frac{4\pi}{c}\boldsymbol{j} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \frac{1}{c}\nabla\left(\frac{\partial\phi}{\partial t}\right) \tag{4.17}$$

Expanding the ∇ terms and use gauge condition to obtain:

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{j} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \nabla(\nabla \cdot \boldsymbol{A}) \tag{4.18}$$

such that

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j} \tag{4.19}$$

Retarded Potentials

Thus, we have to solve

$$\Box^2 \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix} = -4\pi \begin{pmatrix} \rho \\ \boldsymbol{j}/c \end{pmatrix} \tag{4.20}$$

where the d'Alembert operator is

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{4.21}$$

The equations to solve are of the general form

$$\nabla^2\Psi(\boldsymbol{x}, t) - \frac{1}{c^2}\frac{\partial^2\Psi(\boldsymbol{x}, t)}{\partial t^2} = -4\pi f(\boldsymbol{x}, t) \tag{4.22}$$

We will use Green's function method here, based on Jackson, see Shu and Landau&Lifshits for a more elegant way based on physical guessing.

Green's Functions, I

We want to solve

$$\nabla^2\Psi(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\boldsymbol{x},t)}{\partial t^2} = -4\pi f(\boldsymbol{x},t) \qquad (4.22)$$

using the method of Green's functions.

If you have not seen the method of Green's functions before, read the chapters on electrostatics in Jackson. . .

Our aim is to find solutions, G, of

$$\nabla^2 G(\boldsymbol{x},t;\boldsymbol{x}',t') - \frac{1}{c^2}\frac{\partial^2 G(\boldsymbol{x},t;\boldsymbol{x}',t')}{\partial t^2} = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \qquad (4.23)$$

The general solution of Eq. (4.22) is then given by

$$\Psi(\boldsymbol{x},t) = \int_V \int_t G(\boldsymbol{x},t;\boldsymbol{x}',t') f(\boldsymbol{x}',t')\,\mathrm{d}t'\,\mathrm{d}^3x + \begin{pmatrix}\text{solution of homogeneous}\\ \text{version of Eq. (4.22).}\end{pmatrix} \qquad (4.24)$$

Green's Functions, II

To solve wave equations of the type of Eq. (4.22) it is typically better to work in Fourier space

$\Longrightarrow$ Gets rid of the $\partial/\partial t$'s, which unnecessarily complicate life.

Use the Fourier transform as defined by

$$f(\boldsymbol{x},\omega) = \int f(\boldsymbol{x},t)\,\mathrm{e}^{+i\omega t}\,\mathrm{d}t \qquad (4.25)$$

$$G(\boldsymbol{x},\omega;\boldsymbol{x}',t') = \int G(\boldsymbol{x},t;\boldsymbol{x}',t')\,\mathrm{e}^{+i\omega t}\,\mathrm{d}t \qquad (4.26)$$

and the inverse Fourier transform

$$f(\boldsymbol{x},t) = \frac{1}{2\pi}\int f(\boldsymbol{x},\omega)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}\omega \qquad (4.27)$$

$$G(\boldsymbol{x},t;\boldsymbol{x}',t') = \frac{1}{2\pi}\int G(\boldsymbol{x},\omega;\boldsymbol{x}',t')\,\mathrm{e}^{-i\omega t}\,\mathrm{d}\omega \qquad (4.28)$$

Note that the Fourier transform operates on t only, *not* on t'.

Green's Functions, III

Rewrite of the first term of Eq. (4.23) in terms of its Fourier transform:

$$\nabla^2 G(\boldsymbol{x},t;\boldsymbol{x}',t') - \frac{1}{c^2}\frac{\partial^2 G(\boldsymbol{x},t;\boldsymbol{x}',t')}{\partial t^2} = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \qquad (4.23)$$

Green's Functions, V

Rewrite of the second term of Eq. (4.23) in terms of its Fourier transform:

$$\nabla^2 G(\boldsymbol{x},t;\boldsymbol{x}',t') - \frac{1}{c^2}\frac{\partial^2 G(\boldsymbol{x},t;\boldsymbol{x}',t')}{\partial t^2} = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \qquad (4.23)$$

Green's Functions, VII

Rewrite of the right hand side of Eq. (4.23) in terms of its Fourier transform:

$$\nabla^2 G(\boldsymbol{x},t;\boldsymbol{x}',t') - \frac{1}{c^2}\frac{\partial^2 G(\boldsymbol{x},t;\boldsymbol{x}',t')}{\partial t^2} = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \tag{4.23}$$

Green's Functions, IX

Rewriting Eq. (4.23) in terms of its Fourier transform:

$$\nabla^2 G(\boldsymbol{x},t;\boldsymbol{x}',t') - \frac{1}{c^2}\frac{\partial^2 G(\boldsymbol{x},t;\boldsymbol{x}',t')}{\partial t^2} = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \tag{4.23}$$

Collecting the previous results:

$$\nabla^2 G(\boldsymbol{x},t;\boldsymbol{x}',t') = \frac{1}{2\pi}\int \left(\nabla^2 G(\boldsymbol{x},\omega;\boldsymbol{x}',t')\right)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}\omega \tag{4.30}$$

$$-\frac{1}{c^2}\frac{\partial^2 G(\boldsymbol{x},t;\boldsymbol{x}',t')}{\partial t^2} = \frac{1}{2\pi}\int k^2 G(\boldsymbol{x},\omega;\boldsymbol{x}',t')\,\mathrm{e}^{-i\omega t}\,\mathrm{d}\omega \tag{4.33}$$

$$(-4\pi)\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') = \frac{1}{2\pi}\int (-4\pi)\delta(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{e}^{i\omega t'}\,\mathrm{e}^{-i\omega t}\,\mathrm{d}\omega \tag{4.36}$$

Therefore, the Fourier transform of Eq. (4.22) is

$$\left(\nabla^2 + k^2\right) G(\boldsymbol{x},\omega;\boldsymbol{x}',t') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{e}^{i\omega t'} \tag{4.37}$$

Eq. (4.37) is a special case of the Helmholtz wave equation,

$$\left(\nabla^2 + k^2\right)\Psi(\boldsymbol{x},\omega) = -4\pi f(\boldsymbol{x},\omega) \tag{4.38}$$

Green's Functions, X

We will now solve

$$\left(\nabla^2 + k^2\right) G(\boldsymbol{x},\omega;\boldsymbol{x}',t') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{e}^{i\omega t'} \tag{4.37}$$

assuming boundary conditions at infinity.

First note that the right hand side only depends on the radius vector $R := |\boldsymbol{x}-\boldsymbol{x}'| \Longrightarrow$ solution will have to be spherically symmetric.

In spherical coordinates,

$$\nabla_R^2 \bullet = \frac{1}{R}\frac{\mathrm{d}^2}{\mathrm{d}R^2}R\bullet \tag{4.39}$$

such that Eq. (4.37) becomes

$$\frac{1}{R}\frac{\mathrm{d}^2}{\mathrm{d}R^2}\left(RG_k(R)\right) + k^2 G_k(R) = -4\pi\delta(R)\,\mathrm{e}^{i\omega t'} \tag{4.40}$$

where

$$G(\boldsymbol{x},\omega;\boldsymbol{x}',t') =: G_k(R) \tag{4.41}$$

for simplicity.

Green's Functions, XI

First, look at the case $R \neq 0$. Then Eq. (4.40) reads

$$\frac{1}{R}\frac{\mathrm{d}^2}{\mathrm{d}R^2}\left(RG_k(R)\right) + k^2 G_k(R) = 0 \tag{4.42}$$

which is equivalent to

$$\frac{\mathrm{d}^2}{\mathrm{d}R^2}\left(RG_k(R)\right) + k^2\left(RG_k(R)\right) = 0 \tag{4.43}$$

Thus, $RG_k(R)$ obeys the equation of the harmonic oscillator, i.e., its general form is

$$RG_k(R) = A\,\mathrm{e}^{ikR} + B\,\mathrm{e}^{-ikR} \tag{4.44}$$

such that

$$G_k(R) = A\frac{\mathrm{e}^{ikR}}{R} + B\frac{\mathrm{e}^{-ikR}}{R} \tag{4.45}$$

Green's Functions, XII

For $R \neq 0$, the general solution is

$$G_k(R) = A\frac{\mathrm{e}^{ikR}}{R} + B\frac{\mathrm{e}^{-ikR}}{R} \tag{4.45}$$

In order for G_k to be a solution of the general equation,

$$\left(\nabla^2 + k^2\right) G_k(R) = -4\pi\delta(R)\,\mathrm{e}^{i\omega t'} \tag{4.46}$$

we require

$$\lim_{kR\to 0} \nabla_R^2 G_k(R) \longrightarrow -4\pi\delta(R)\,\mathrm{e}^{i\omega t'} \tag{4.47}$$

Since $\lim_{kR\to 0}\exp(ikR) = 1$, it is sufficient to look at $\nabla_R^2(1/R)$.

Because of the Poisson identity

$$\nabla^2\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right) = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}') \quad\Longrightarrow\quad \nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(R) \tag{4.48}$$

This is obtained, e.g., from considering the total charge enclosed in a volume around a point source.

This means that $G_k(R)$ is a solution of Eq. (4.37) if

$$A + B = \mathrm{e}^{i\omega t'} \tag{4.49}$$

Advanced and Retarded Green's Function, I

Now set $A = 0$ or $B = 0$, this gives ingoing and outgoing waves,

$$G^{\pm}(\boldsymbol{x},\omega;\boldsymbol{x}',t') = \frac{\mathrm{e}^{\pm ikr}}{R}\,\mathrm{e}^{i\omega t'} \tag{4.50}$$

Go back to real space by inverse Fourier transforming (Eq. 4.25):

$$G^{\pm}(\boldsymbol{x},t;\boldsymbol{x}',t') = \frac{1}{2\pi}\int \frac{\mathrm{e}^{\pm ikr}}{R}\,\mathrm{e}^{i\omega t'}\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t \tag{4.51}$$

insert $k = \omega/c$,

$$= \frac{1}{2\pi R}\int \mathrm{e}^{\pm i\omega\frac{R}{c} - i\omega(t-t')}\,\mathrm{d}t \tag{4.52}$$

remembering the definition of the δ-function, Eq. (4.34)

$$= \frac{1}{R}\cdot\delta\left((t-t') \mp \frac{R}{c}\right) \tag{4.53}$$

such that finally

$$G^{\pm}(\boldsymbol{x},t;\boldsymbol{x}',t') = \delta\left(t' - \left[t \pm \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right]\right)\cdot\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{4.54}$$

Advanced and Retarded Green's Function, II

The Green's functions of Eq. (4.22) are

$$G^{\pm}(\boldsymbol{x},t;\boldsymbol{x}',t') = \delta\left(t' - \left[t \pm \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right]\right)\cdot\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{4.54}$$

- G^-: retarded Green's function: effect of a source at distance R observed at $(\boldsymbol{x},t)$ is due to source behavior at time $t' = t - R/c$.
- G^+: advanced Green's function: effect of a source at distance R observed at $(\boldsymbol{x},t)$ is due to source behavior at time $t' = t + R/c$ in the future!

Sommerfeld: advanced Green's function violates causality $\Longrightarrow$ Ignore!

Feynman: G^+ has applications, e.g., in quantum electrodynamics $\Longrightarrow$ not total crap...

Will only use G^- from now on.

Liénard-Wiechert Potentials

We now use the retarded Green's function in Eq. (4.20) to obtain solutions for ϕ, $\boldsymbol{A}$:

$$\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} = \int_V \begin{pmatrix}\rho(\boldsymbol{x}',t')\\ \boldsymbol{j}(\boldsymbol{x}',t')/c\end{pmatrix}\frac{\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{4.55}$$

where the retarded time

$$t' = t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c} \tag{4.56}$$

For a moving particle,

$$\rho(t) = q\delta(\boldsymbol{x}-\boldsymbol{r}(t)) \tag{4.57}$$

$$\boldsymbol{j}(t) = q\boldsymbol{v}\delta(\boldsymbol{x}-\boldsymbol{r}(t)) \tag{4.58}$$

Inserting into Eq. (4.55) gives after using some tricks the Liénart-Wiechert potentials:

$$\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} = \left[\frac{1}{R - \boldsymbol{R}\cdot\boldsymbol{v}/c}\cdot\begin{pmatrix}q\\ q\boldsymbol{j}/c\end{pmatrix}\right]_{\text{ret}} \tag{4.59}$$

evaluated at the retarded time $\tau = 0$, which is found by solving

$$t = \tau + R(\tau)/c \quad\text{where}\quad \boldsymbol{R}(\tau) = \boldsymbol{x} - \boldsymbol{r}(\tau) \tag{4.60}$$

$\boldsymbol{v} = 0$: $\phi = q/R$, $\boldsymbol{A} = 0 \Longrightarrow$ Coulomb!

$\boldsymbol{v} \neq 0$: $\boldsymbol{A} \neq 0 \Longrightarrow \boldsymbol{B} \neq 0 \Longrightarrow$ Moving charge produces $\boldsymbol{B}$-field!

Quite a few "tricks" are needed to derive the Liénart-Wiechert potentials. We start with the moving particle:

$$\rho(t) = q\delta(\boldsymbol{x} - \boldsymbol{r}(t)) \quad (4.57)$$
$$\boldsymbol{j}(t) = q\boldsymbol{v}\delta(\boldsymbol{x} - \boldsymbol{r}(t)) \quad (4.58)$$

These equations can also be made to look more complicated and to read

$$\begin{pmatrix}\rho(\boldsymbol{x}',t')\\ \boldsymbol{j}(\boldsymbol{x}',t')/c\end{pmatrix} = \int \begin{pmatrix}q\\ q\boldsymbol{v}(\tau)/c\end{pmatrix}\cdot\delta(\boldsymbol{x}'-\boldsymbol{r}(\tau))\cdot\delta(\tau-t')\,\mathrm{d}\tau \quad (4.61)$$

Thus, inserting into Eq. (4.55),

$$\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} = \int_V\int_\tau \begin{pmatrix}q\\ q\boldsymbol{v}(\tau)/c\end{pmatrix}\cdot\frac{\delta(\boldsymbol{x}'-\boldsymbol{r}(\tau))\delta(\tau-t')\,\mathrm{d}\tau\,\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} \quad (4.62)$$
$$= \int_\tau \begin{pmatrix}q\\ q\boldsymbol{v}(\tau)/v\end{pmatrix}\frac{\delta(\tau-t')\,\mathrm{d}\tau}{|\boldsymbol{x}-\boldsymbol{r}(\tau)|} \quad (4.63)$$

where

$$t' = t - |\boldsymbol{x}-\boldsymbol{r}(\tau)|/c =: t - R(\tau)/c \quad (4.64)$$

Thus, to get ϕ and $\boldsymbol{A}$, we integrate over the past history of the particle. This is nice, however, there is one problem since there is an implicit dependence on the path in the δ-function. To simplify the equation further, substitute

$$\tau' = \tau - t' = \tau - t + R(\tau)/c \quad (4.65)$$

and therefore

$$\frac{\mathrm{d}\tau'}{\mathrm{d}\tau} = 1 + \frac{\dot{R}(\tau)}{c} \quad\Longrightarrow\quad \mathrm{d}\tau' = \left(1+\frac{\dot{R}(\tau)}{c}\right)\mathrm{d}\tau \quad (4.66)$$

such that

$$\frac{\mathrm{d}\tau}{|\boldsymbol{x}-\boldsymbol{r}(\tau)|} = \frac{\mathrm{d}\tau}{|\boldsymbol{R}(\tau)|} = \frac{\mathrm{d}\tau'}{|\boldsymbol{R}(\tau)|(1+\dot{R}(\tau)/c)} = \frac{\mathrm{d}\tau'}{R(\tau)(1+\dot{R}(\tau)/c)} = \frac{\mathrm{d}\tau'}{R(\tau)+R(\tau)\dot{R}(\tau)/c)} \quad (4.67)$$

But since

$$\boldsymbol{R}(\tau) = \boldsymbol{x}-\boldsymbol{r}(\tau) \quad\Longrightarrow\quad \dot{\boldsymbol{R}}(\tau) = -\dot{\boldsymbol{r}}(\tau) = -\boldsymbol{v}(\tau) \quad (4.68)$$

Such that

$$\frac{\mathrm{d}\tau}{|\boldsymbol{x}-\boldsymbol{r}(\tau)|} = \frac{\mathrm{d}\tau'}{R(\tau)-\boldsymbol{R}(\tau)\cdot\boldsymbol{v}(\tau)/c} \quad (4.69)$$

Note the boosting by the $\boldsymbol{R}\cdot\boldsymbol{c}$ term!

Inserting into Eq. (4.63) then gives

$$\begin{pmatrix}\phi(\boldsymbol{x},t)\\ \boldsymbol{A}(\boldsymbol{x},t)\end{pmatrix} = \int\begin{pmatrix}q\\ q\boldsymbol{v}/c\end{pmatrix}\cdot\frac{\delta(\tau')\,\mathrm{d}\tau'}{R(\tau)-\boldsymbol{R}(\tau)\cdot\boldsymbol{v}(\tau)/c} \quad (4.70)$$

Evaluating the δ function then finally gives the Liénard-Wiechert potentials.

Calculation of Field, I

To obtain the EM fields from ϕ and $\boldsymbol{A}$, insert Liénart-Wiechert into Eqs. (4.10) and (4.12),

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} + \nabla\phi \qquad\text{and}\qquad \boldsymbol{B} = \nabla\times\boldsymbol{A} \quad (4.10,4.12)$$

Calculation is in principle straightforward, although somewhat tedious $\Longrightarrow$ See Jackson for details.

Calculation involves going from τ to t, i.e., one has to compute quantities such as

$$\frac{\partial\tau}{\partial t} = \left(1 - \frac{1}{c}\frac{\boldsymbol{R}}{R}\cdot\frac{\boldsymbol{v}}{c}\right) \quad (4.71)$$

i.e., this is ratio of intervals between the two "same" events, $\Delta\tau$ as experienced by particle, and Δt as seen by observer (see Eq. 4.67ff.).

The result is

$$\boldsymbol{E} = \frac{q}{(R-\boldsymbol{R}\cdot\boldsymbol{\beta})^3}\left\{(1-\beta^2)(\boldsymbol{R}-R\boldsymbol{\beta}) + \frac{\boldsymbol{R}}{c}\times\left[(\boldsymbol{R}-R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\} \quad (4.72)$$

$$\boldsymbol{B} = \frac{\boldsymbol{R}}{R}\times\boldsymbol{E} \quad (\text{note: } \boldsymbol{B}\perp\boldsymbol{E}!) \quad (4.73)$$

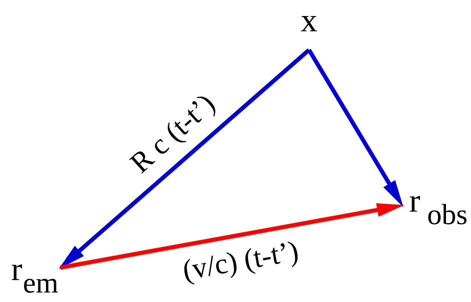
where

$$\boldsymbol{\beta} = \boldsymbol{v}/c \quad (4.74)$$

Calculation of Field, II

The equation for $\boldsymbol{E}$, Eq. (4.72), has two components:

1. First summand, $\propto R^{-2}$, depending on velocity: velocity field or Coulomb part.
2. Second summand, $\propto R^{-1}$, depends on acceleration: acceleration field or radiation field



For a particle moving in a straight line with constant velocity, the $\boldsymbol{E}$ field always points towards the *current* position of the particle:

1. Position of particle at "emission": $\propto \boldsymbol{R}(\tau)c(t-\tau)$.
2. Motion of particle between "emission" and "observation": $\propto \boldsymbol{v}(t-\tau)$.

$\Longrightarrow$ Displacement between "emission" and "observation" points: $\propto (\boldsymbol{R}(\tau)-\boldsymbol{v})(t-\tau)$ $\Longrightarrow \propto \boldsymbol{E}$.

Nonrelativistic motion, I

Interpretation of Eq. (4.72):

$$\boldsymbol{E} = \frac{q}{(R - \boldsymbol{R}\cdot\boldsymbol{\beta})^3}\left\{(1-\beta^2)(\boldsymbol{R} - R\boldsymbol{\beta}) + \frac{\boldsymbol{R}}{c}\times\left[(\boldsymbol{R} - R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\} \quad (4.72)$$

In the nonrelativistic regime, i.e., where $|\boldsymbol{\beta}| = \frac{v}{c} \ll 1$, the magnitude of velocity field is

$$E_{\text{vel}} \sim \frac{1}{R^3}\cdot R = \frac{1}{R^2} \quad (4.75)$$

and magnitude of radiation field is

$$E_{\text{rad}} \sim \frac{1}{R^3}\cdot\frac{R}{c}\cdot R\cdot\dot{\beta} = \frac{\dot{v}}{Rc^2} \quad (4.76)$$

such that

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} \sim \frac{R\dot{v}}{c^2} \quad (4.77)$$

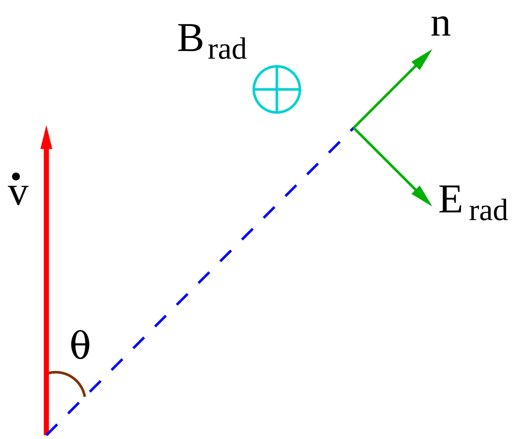
Assume particle has characteristic oscillation frequency, ν. Then, $\dot{v} \sim v\nu$ and

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} \sim \frac{Rv\nu}{c^2} = \frac{v}{c}\frac{R}{\lambda} \quad (4.78)$$

- $R \lesssim \lambda$: E_{vel} dominates E_{rad} by $\sim c/u \Longrightarrow$ "near zone".
- $R \gtrsim \lambda(v/c)^{-1}$: $E_{\text{rad}} \gg E_{\text{vel}} \Longrightarrow$ "far zone" or "wave zone".

In general, astronomical objects are far away and "far zone" approximation is sufficient. . .

Nonrelativistic motion, II



Assume far zone. From Eq. (4.72), the radiation field is

$$\boldsymbol{E}_{\text{rad}} = \frac{q}{(R - \boldsymbol{R}\cdot\boldsymbol{\beta})^3}\frac{\boldsymbol{R}}{c}\times \times\left[(\boldsymbol{R} - R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right] \quad (4.79)$$

assume $\beta \ll 1$,

$$\boldsymbol{E}_{\text{rad}} = \frac{q}{Rc^2}\boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{v}}) \quad (4.80)$$

$$\boldsymbol{B}_{\text{rad}} = \boldsymbol{n}\times\boldsymbol{E}_{\text{rad}} \quad (4.81)$$

where $\boldsymbol{n} = \boldsymbol{R}/R$. Magnitude of $\boldsymbol{E}$, $\boldsymbol{B}$:

$$E_{\text{rad}} = B_{\text{rad}} = \frac{q\dot{v}}{Rc^2}\sin\theta \quad (4.82)$$

where $\theta = \angle(\dot{\boldsymbol{v}}, \boldsymbol{n})$.

Larmor's formula

To calculate emitted power, look at Poynting vector, $\boldsymbol{S}$, which is in the direction of $\boldsymbol{n}$ and has the magnitude (Eq. 2.33)

$$S = \frac{c}{4\pi}E_{\text{rad}}B_{\text{rad}} = \frac{c}{4\pi}E_{\text{rad}}^2 = \frac{c}{4\pi}\frac{q^2\dot{v}^2}{R^2c^4}\sin^2\theta \quad (4.83)$$

This is the energy flow in direction $\boldsymbol{n}$ (unit: $\mathrm{erg\,s^{-1}\,cm^{-2}}$)

To obtain power, multiply with area $\mathrm{d}A = R^2\,\mathrm{d}\Omega$:

$$\frac{\mathrm{d}W}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{q^2\dot{v}^2}{4\pi c^3}\sin^2\theta \quad (4.84)$$

The total power is given by Larmor's formula:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{q^2\dot{v}^2}{4\pi c^3}\int\sin^2\theta\,\mathrm{d}\Omega = \frac{2q^2}{3c^3}\dot{v}^2 \quad (4.85)$$

Note

1. Power is $\propto q^2\dot{v}^2$
2. note the Dipole pattern, $\propto \sin^2\theta \Longrightarrow$ no radiation emitted $\parallel \boldsymbol{v}$, maximum radiation emitted $\perp \boldsymbol{v}$.
3. Direction of $\boldsymbol{E}_{\text{rad}}$ determined by $\dot{\boldsymbol{v}}$ and $\boldsymbol{n}$. For acceleration along a line: Radiation linearly polarized in plane of $\dot{\boldsymbol{v}}$ and $\boldsymbol{n}$.

Dipole Approximation, I

Last thing to do is to think about radiation from many particles. In principle trivial, just add $\boldsymbol{E}_{\text{rad}}$ from all particles.

BUT

All equations used retarded times $\Longrightarrow$ different for each particle! $\Longrightarrow$ need to keep track of phase relations between different particles $\Longrightarrow$ highly nontrivial. . .

Can ignore these problems, though:

Assume system size L and typical time scale for emission $\tau \Longrightarrow$ Differences in retarded time within the system are negligible if $\tau \gg L/c$ (light travel time).

Alternatively, consider frequency $\nu \sim 1/\tau$. Then

$$\tau \gg L/c \quad \Longleftrightarrow \quad \frac{c}{\nu} = \lambda \gg L \quad (4.86)$$

i.e., if a system is smaller than its emitted wavelength, we can ignore its size.

Note that $\tau \sim v/l$ where $l \ll L$ is a characteristic size of system. Thus $v/c \ll l/L$, i.e., $v \ll c \Longrightarrow$ can use nonrelativistic formulae throughout.

Dipole Approximation, II

Thus, for an ensemble of particles,

$$\boldsymbol{E}_{\text{rad}} = \sum_i \frac{q_i}{c^2} \frac{\boldsymbol{n} \times (\boldsymbol{n} \times \dot{\boldsymbol{v}}_i)}{R_i} \tag{4.87}$$

For large distance, R, from system, $R_i \sim R$ and

$$\boldsymbol{E}_{\text{rad}} = \sum_i \frac{1}{c^2} \frac{\boldsymbol{n} \times (\boldsymbol{n} \times q_i \dot{\boldsymbol{v}}_i)}{R} \tag{4.88}$$

$$= \frac{1}{c^2} \frac{\boldsymbol{n} \times (\boldsymbol{n} \times \sum_i q_i \dot{\boldsymbol{v}}_i)}{R} \tag{4.89}$$

$$= \frac{1}{c^2} \frac{\boldsymbol{n} \times (\boldsymbol{n} \times \ddot{\boldsymbol{d}})}{R} \tag{4.90}$$

where

$$\boldsymbol{d} = \sum_i q_i \boldsymbol{r}_i \tag{4.91}$$

is the dipole moment. Therefore, as before

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\ddot{\boldsymbol{d}}^2}{4\pi c^3} \sin^2\theta \quad \text{and} \quad P = \frac{2\ddot{\boldsymbol{d}}^2}{3c^3} \tag{4.92}$$

the dipole approximation.