Friedrich-Alexander-Universität
Erlangen-Nürnberg

Astrophysical Radiation Processes

Jörn Wilms

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Sommersemester 2008

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Introduction

Schedule

Introduction
14.04. Introduction, Multiwavelength Astrophysics
21.04. Radiation and Radiative Transfer
28.04. Black Body Radiation
Classical Radiation Theory
05.05. Radiation from Moving Charges
12.05. *No lecture – Pentacost*
19.05. Bremsstrahlung
26.05. Synchrotron Radiation
02.06. Comptonization
09.06. Pair Production
16.06. Radiation from Nuclei
Atomic (Quantum-Mechanical) Processes
23.06. Atomic Structure
30.06. Line Diagnostics
07.07. Molecular Radiation
14.07. *No lecture (conference)*

Literature

RYBICKI, G.B. & LIGHTMAN, A.P., 1979, *Radiative Processes in Astrophysics*, New York: Wiley, $116

A "must buy", although now very expensive (I got it for $50). Standard text of the field, in some areas getting outdated, though – get it from amazon.com, *not* amazon.de

PADMANABHAN, T., 2000, Theoretical Astrophysics: Volume 1: Astrophysical Processes, Cambridge: Cambridge Univ. Press, 65.00€

Introduction to the physics of astrophysics. Short, concise, great.

PADMANABHAN, T., 2006, An Invitation to Astrophysics, New Jersey: World Scientific, $36.00

A beautifully written overview of the major physical processes relevant for astrophysics (not only graviation).

Literature

LONGAIR, M.S., 1992, *High Energy Astrophysics, Vol. 1: Particles, Photons, and their Detection*, Cambridge: Cambridge Univ. Press, ~50€

Good introduction to high energy astrophysics, the 1st volume deals extensively with high energy procsses. Recommended. Unfortunately, everything is in SI units.

SHU, F.H., 1991, *The Physics of Astrophysics, I. Radiation*, Mill Valley: University Science Books, 70.00€

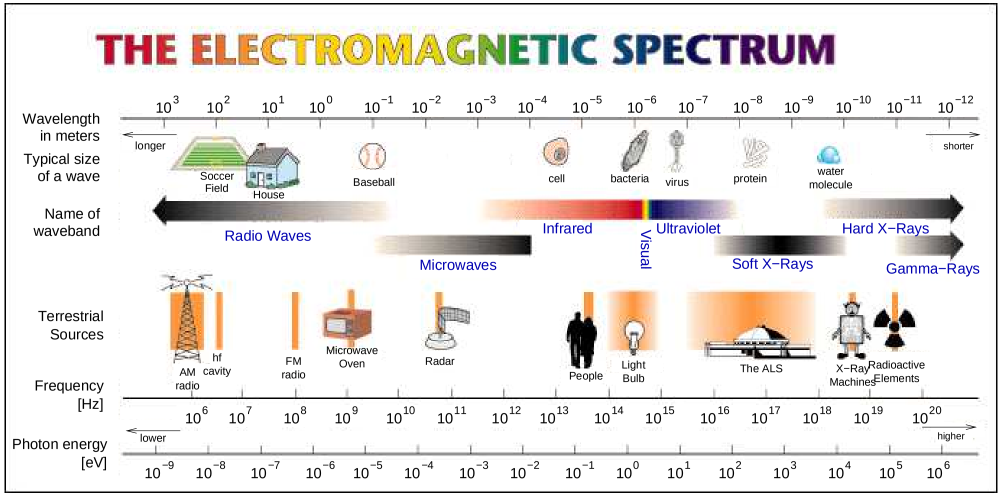
Good introduction to radiation processes, some important areas are missing, though. Not as understandable as Rybicki & Lightman.

LANG, K.R., 1999, Astrophysical Formulae, 3rd edition, 2 Vols, Heidelberg: Springer, 2×107€

Collection of 1000s of formulae necessary for astrophysical research, with exhaustive references to the original literature.

COWLEY, C.R., 1995, An Introduction to Cosmochemistry, Cambridge: Cambridge Univ. Press, $37

Practical summary of atomic and molecular processes.



Literature

Good references on the WWW on radiation processes include

Ernie Seaquist, University of Toronto, AST1440F: Radiation Processes

Lecture notes on an advanced lecture on astrophysical radiation processes, approximately at the level of this lecture, available at `http://www.astro.utoronto.ca/~seaquist/radiation/notes.html`.

Juri Poutanen, University of Oulu, Radiative Processes in Astrophysics 2006

More formal than this course, good introduction on relativistic mechanisms, which we will ignore for time reasons, available at `http://cc.oulu.fi/~jpoutane/teaching/rad06.html`.

Jelle Kaastra et al., Thermal Radiation Processes, Space Sci. Rev., submitted

Well written overview of the atomic processes which are relevant for the computation of photoionized plasmas (an area which we will not be able to discuss in detail due to time reasons); available at `http://arxiv.org/abs/0801.1011`.

Electromagnetic Spectrum, II

As we all know, light can be characterized by

Wavelength: λ, measured in m, mm, cm, nm, Å.

Frequency: ν, measured in Hz, MHz.

Energy: E, measured in J, erg, Rydbergs, eV, keV, MeV, GeV.

Temperature: T, measured in K.

These quantities are related:

$$\lambda\nu = c \qquad E = h\nu \qquad T = E/k \tag{1.1}$$

where

$$c = 299792458\,\mathrm{m\,s^{-1}} \sim 2.998\times10^{10}\,\mathrm{cm\,s^{-1}} \tag{1.2}$$

$$h = 6.6260693(11)\times10^{-34}\,\mathrm{J\,s} \sim 6.626\times10^{-27}\,\mathrm{erg\,s} \tag{1.3}$$

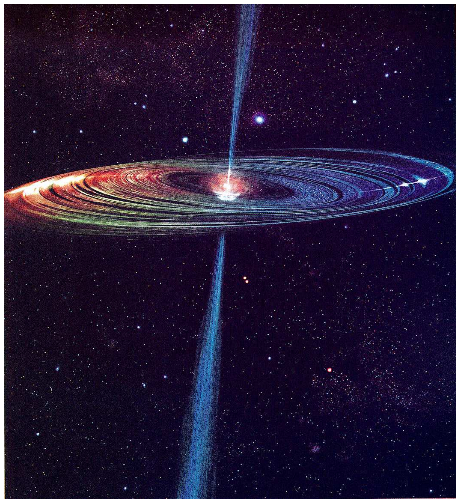
$$k = 1.3806505(24)\times10^{-23}\,\mathrm{J\,K^{-1}} \sim 1.38\times10^{-16}\,\mathrm{erg\,K^{-1}} \tag{1.4}$$

Constants are 2002 CODATA values, `http://physics.nist.gov/cuu/Constants/index.html` uncertainty is 1σ in units of last digit shown.

Conversion table (courtesy Eureka Scientific, `www.eurekasci.com`):

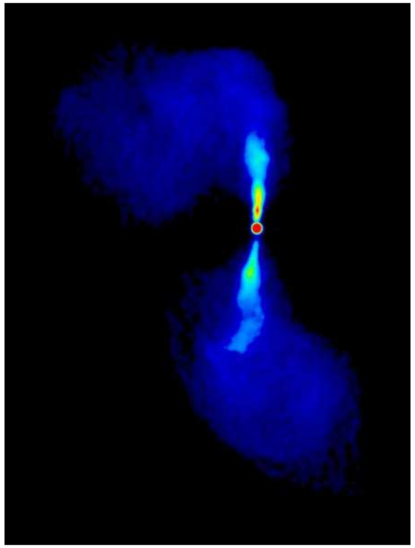
From ⇓ To ⟹	λ[Å]	λ[μm]	λ[cm]	ν[Hz]	E[keV]	E[erg]
λ[Å]	1	$10^{-4}\lambda$	$10^{-8}\lambda$	$3\times10^{18}/\lambda$	$12.4/\lambda$	$2\times10^{-8}/\lambda$
λ[μm]	$10^{4}\lambda$	1	$10^{-4}\lambda$	$3\times10^{14}/\lambda$	$1.24\times10^{-3}/\lambda$	$2\times10^{-12}/\lambda$
λ[cm]	$10^{8}\lambda$	$10^{4}\lambda$	1	$3\times10^{10}/\lambda$	$1.24\times10^{-7}/\lambda$	$2\times10^{-16}/\lambda$
ν[Hz]	$3\times10^{18}/\nu$	$3\times10^{14}/\nu$	$3\times10^{10}/\nu$	1	$4.14\times10^{-18}\nu$	$6.63\times10^{-27}\nu$
E[keV]	$12.4/E$	$1.24\times10^{-3}/E$	$1.24\times10^{-7}/E$	$2.42\times10^{17}E$	1	$1.60\times10^{-9}E$
E[erg]	$2\times10^{-8}/E$	$2\times10^{-12}/E$	$2\times10^{-16}/E$	$1.51\times10^{26}E$	$6.24\times10^{8}E$	1

Why Multiwavelength Astronomy?, I



Structure of Active Galactic Nuclei (AGN):
- supermassive black hole ($10^7\,M_\odot$)
- accretion disk ($\dot{M}\sim 1\ldots 2\,M_\odot\,\mathrm{yr}^{-1}$)
- large luminosity ($L\sim 10^{10}\,L_\odot$)
- Schwarzschild radius $2GM/c^2\sim 1$ AU

Why Multiwavelength Astronomy?, II



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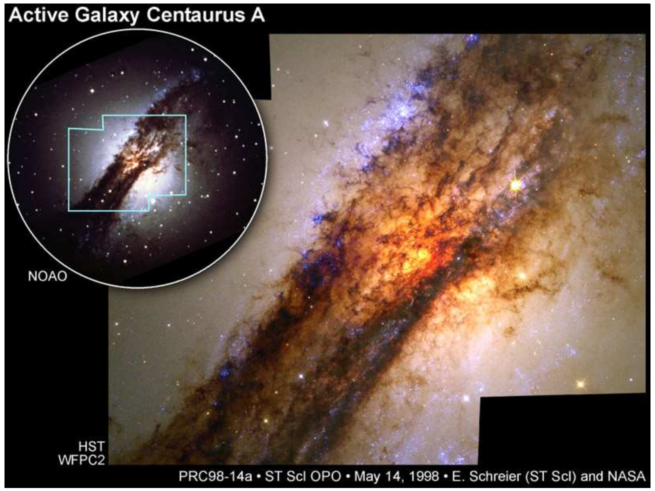
- Schwarzschild radius $2GM/c^2\sim 1$ AU
- often relativistic jets, where material is accelerated to the speed of light

AGN *with* jets: quasars, blazars...
AGN *without* jets: Seyfert galaxies



In the following as an example: Centaurus A (NGC 5128)
- one of the brightest radio sources in the sky
- distance: 11 million light years
- giant elliptical galaxy (more properly: S0), merged with spiral galaxy about 100 million years ago, remnant of the spiral seen as dust lane.

AGN are exceptionally good examples for the importance of multi-wavelength astronomy.

Centaurus A

Optical:
Thermal emission from stars and gas, i.e., bremsstrahlung (free-free radiation), line emission, dust scattering,. . .)

Cen A: VLT Kueyen+FORS2, courtesy ESO

Centaurus A

Near Infrared:
Thermal emission, mainly from stars, similar to optical, but dust less apparent
$\Longrightarrow$ *Opacity* of dust in IR is smaller.

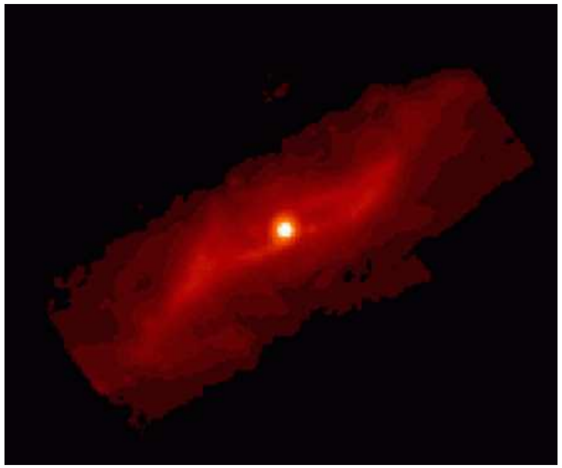
2MASS, courtesy IPAC, Univ. Massachusetts

Centaurus A

Mid Infrared (3.6–8 μm):
Thermal emission from dust starts to dominate, contribution of thermal emission from stars still significant.

Spitzer Space Telescope, courtesy Caltech/NASA

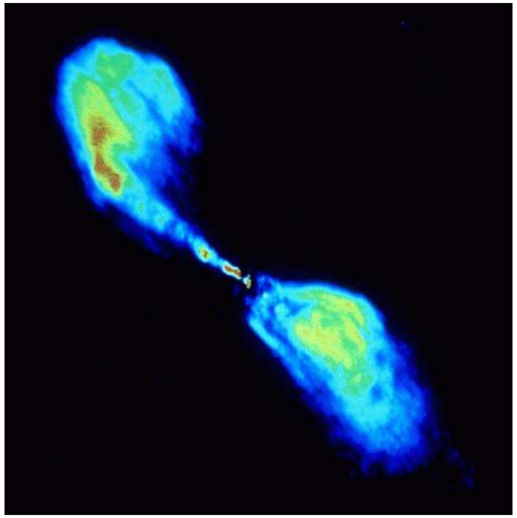
Centaurus A



Far Infrared (7 μm):
Thermal emission from dust
Resolution of this image is worse than the previous Spitzer telescope image.

ISO, courtesy ESA-ESTEC

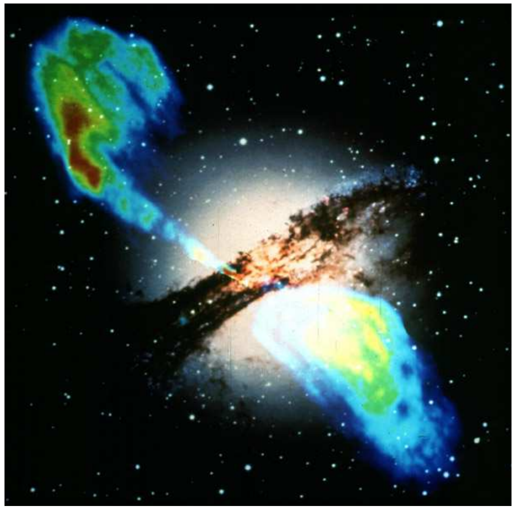
Centaurus A



Radio (6 cm):
Synchrotron radiation from jets and black hole.

VLA, courtesy NRAO

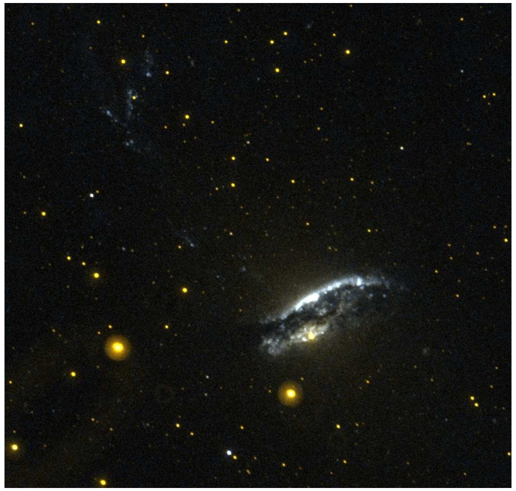
Centaurus A



Radio (6 cm):
Synchrotron radiation from jets and black hole.

VLA/optical, courtesy STScI

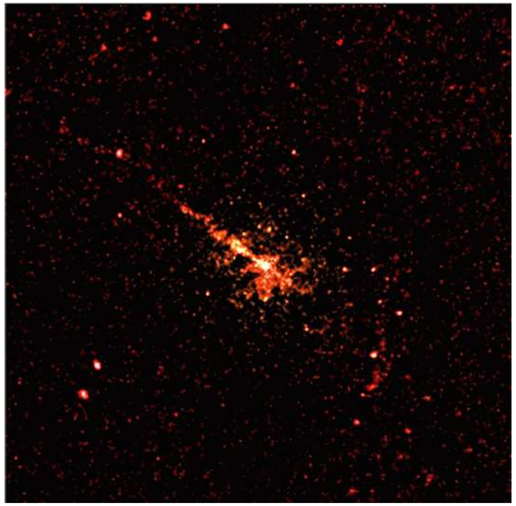
Centaurus A



UV (30–300 nm):
Thermal UV emission from young stars (in NE corner)
Photoabsorption and absorption by dust by dust lane

GALEX, courtesy NASA/Caltech

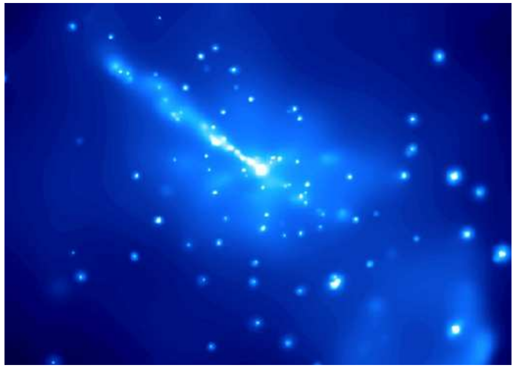
Centaurus A



X-rays (2–10 keV):
- Synchrotron radiation from jet,
- Comptonized photons from black hole,
- other emission from X-ray binaries and background AGN

Chandra, courtesy CXC

Centaurus A



Chandra, courtesy CXC

X-rays (2–10 keV):

- Synchrotron radiation from jet,
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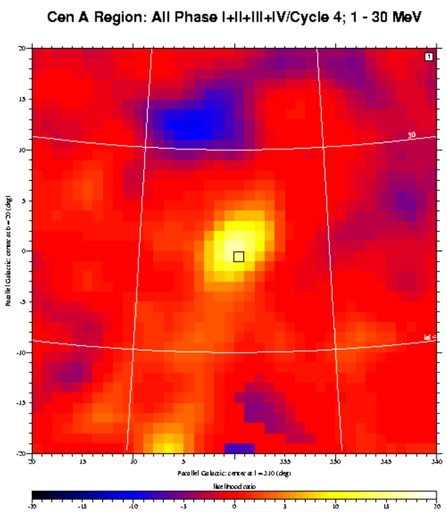
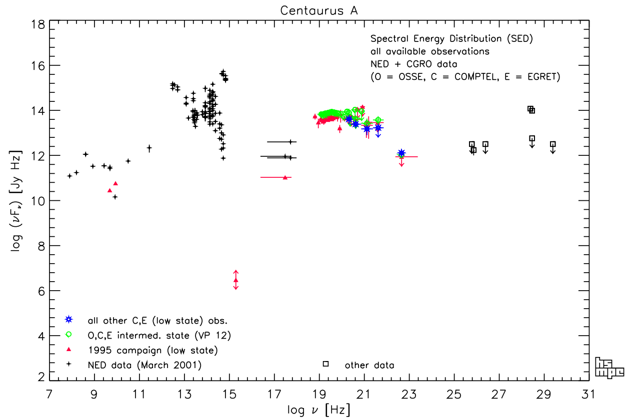
Centaurus A



CGRO-COMPTEL, courtesy MPE/H. Steinle

γ-rays (1–30 MeV):
Comptonized synchrotron radiation from jet and/or black hole.

Centaurus A



Steinle (2006, Chin. J. Astron. Astrophys. 6(Suppl. 1), 106)

Broad-band spectrum of Cen A: Spectral Energy Distribution (SED) νf_ν is flat
$\Longrightarrow$ similar energy output at all wavebands!

Shown is a νf_ν-plot, where ν: frequency, f_ν: flux density at frequency ν (units of f_ν are erg s^{-1} cm^{-2} Hz^{-1}). Since $\int_{\nu_1}^{\nu_2} \nu f_\nu d\nu = \int_{\ln \nu_1}^{\ln \nu_2} f_\nu d \ln \nu$ plotting νf_ν in a log-log-plot gives a measure of the energy emitted per frequency decade.

Radiation and Radiative Transfer

Maxwell's Equations

In classical electrodynamics, electromagnetism is described by Maxwell's equations:

Coulomb's law:

$$\nabla \cdot \boldsymbol{E} = \left[\frac{1}{4\pi\epsilon_0}\right] 4\pi\rho \qquad (2.1)$$

Law of Induction (Faraday):

$$\nabla \times \boldsymbol{E} = -[c]\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad (2.2)$$

Nonexistence of Magnetic Monopoles (Gilbert):

$$\nabla \cdot \boldsymbol{B} = 0 \qquad (2.3)$$

Ampère's law:

$$\nabla \times \boldsymbol{B} = \left[\frac{c\mu_0}{4\pi}\right]\frac{4\pi}{c}\boldsymbol{j} + \left[\frac{1}{c}\right]\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \qquad (2.4)$$

Where current, $\boldsymbol{j}$, and charge density, ρ, are coupled by the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \qquad (2.5)$$

To derive the wave equations, look at the Maxwell equations in vacuum:

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \qquad (2.7)$$

Therefore

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad (2.9)$$

But

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} \qquad (2.10)$$

and furthermore $\nabla \cdot \boldsymbol{E} = 0$ in vacuum. Therefore

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2\nabla^2 \boldsymbol{E} = 0 \qquad (2.8)$$

The wave equation for $\boldsymbol{B}$ is obtained in a similar manner (since Maxwell's equations are invariant with respect to the change $\boldsymbol{E} \to \boldsymbol{B}$ and $\boldsymbol{B} \to -\boldsymbol{E}$).

EM Waves, I

One of the most important solutions to Maxwell's equations is the electromagnetic wave in vacuum.

Vacuum means:

$$\rho = 0 \quad \text{and} \quad \boldsymbol{j} = 0 \qquad (2.6)$$

In cgs ("Gaussian") units, Maxwell's equations then become

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \qquad (2.7)$$

As shown on the handout, these are equivalent to the vacuum wave equations:

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2\nabla^2 \boldsymbol{E} = 0 \quad \text{and} \quad \frac{\partial^2 \boldsymbol{B}}{\partial t^2} - c^2\nabla^2 \boldsymbol{B} = 0 \qquad (2.8)$$

The equations are homogeneous wave equations.

1. Note that Maxwell for $\partial\boldsymbol{E}/\partial t$ and $\partial\boldsymbol{B}/\partial t$ imply $\boldsymbol{E} \perp \boldsymbol{B}$.
2. General solutions usually obtained using Fourier transforms.

EM Waves, II

One possible solution to the wave equations

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \boldsymbol{B} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} \qquad (2.8)$$

is given by

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{a}_1 E_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \quad \text{and} \quad \boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{a}_2 B_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \qquad (2.11)$$

where $\boldsymbol{a}_{1,2}$ are unit vectors specifying the directions of $\boldsymbol{E}$ and $\boldsymbol{B}$, and the wave vector, $\boldsymbol{k}$ is related to the frequency, ν, and wavelength, λ, via

$$|\boldsymbol{k}| = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} = \frac{\omega}{c} \qquad (2.12)$$

The solutions to Eq. 2.8 are plane waves traveling in the direction of $\boldsymbol{k}$.

See handout for proof.

Plugging the wave equations into Maxwell's equations allows us to understand a few further properties of electromagnetic radiation, which will be of great use later on.

In order to make progress, we need the following vector identities:

$$\nabla \cdot (f\boldsymbol{a}) = f(\nabla \cdot \boldsymbol{a}) + (\nabla f) \cdot \boldsymbol{a} = (\nabla f) \cdot \boldsymbol{a} \quad (2.13)$$

$$\nabla \times (f\boldsymbol{a}) = f(\nabla \times \boldsymbol{a}) + (\nabla f) \times \boldsymbol{a} = (\nabla f) \times \boldsymbol{a} \quad (2.14)$$

where the 2nd equation holds for $\boldsymbol{a} = \text{const.}$.

For a plane wave, the above equations give (note that ∇ works on $\boldsymbol{r}$ only!)

$$\nabla \cdot \boldsymbol{E} = \nabla \left(E_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \right) \cdot \boldsymbol{a}_1 = -iE_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}(\boldsymbol{k} \cdot \boldsymbol{a}_1) \quad (2.15)$$

But in vacuum, $\rho = 0$ and therefore Coulomb's law (Eq. (2.1)) gives

$$\nabla \cdot \boldsymbol{E} = 0 \quad (2.16)$$

and therefore

$$\boldsymbol{k} \cdot \boldsymbol{a}_1 = 0 \quad (2.17)$$

A similar calculation for $\boldsymbol{B}$ shows that

$$\boldsymbol{k} \cdot \boldsymbol{a}_2 = 0 \quad (2.18)$$

Furthermore,

$$\nabla \times \boldsymbol{E} = \nabla \left(E_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \right) \times \boldsymbol{a}_1 = -iE_0 e^{i(\omega t - \boldsymbol{k}\times\boldsymbol{r})}(\boldsymbol{k} \times \boldsymbol{a}_1) \quad (2.19)$$

But because of Faraday's law of induction (Eq. 2.2),

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{a}_2 B_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \right) = -\frac{i\omega}{c} B_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \boldsymbol{a}_2 \quad (2.20)$$

and therefore

$$E_0(\boldsymbol{k} \times \boldsymbol{a}_1) = \frac{\omega}{c} B_0 \boldsymbol{a}_2 \quad (2.21)$$

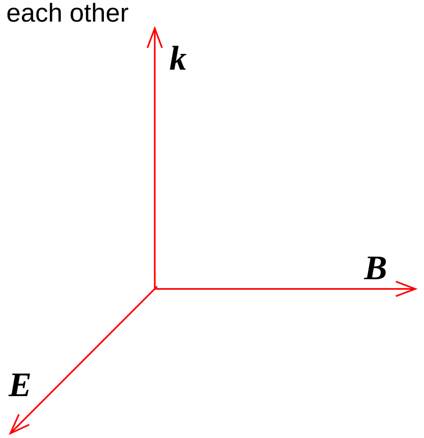
and analoguously

$$B_0(\boldsymbol{k} \times \boldsymbol{a}_2) = -\frac{\omega}{c} B_0 \boldsymbol{a}_1 \quad (2.22)$$

EM Waves, III

As shown on the handout:

- $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{k}$ are orthogonal to each other



- $E_0 = \omega B_0/kc$ and $B_0 = \omega E_0/kc$, such that

$$E_0 = \left(\frac{\omega}{kc}\right)^2 E_0 \quad (2.23)$$

or

$$\omega^2 = c^2 k^2 \quad (2.24)$$

But since $k = 2\pi/\lambda = 2\pi\nu/c$

$$\lambda\nu = c \quad (2.25)$$

- Because of the above, in Gaussian units

$$E_0 = B_0 \quad (2.26)$$

Continuity and Lorentz Force

Assume all charges are point charges, then

$$\rho(\boldsymbol{x}, t) = \sum_i q_i \delta(\boldsymbol{x} - \boldsymbol{x}_i(t)) \quad (2.27)$$

$$\boldsymbol{j}(\boldsymbol{x}, t) = \sum_i q_i \boldsymbol{v}_i \delta(\boldsymbol{x} - \boldsymbol{x}_i(t)) \quad (2.28)$$

Positions $\boldsymbol{x}_i(t)$ are computed from

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i \quad \text{and} \quad \frac{d\boldsymbol{p}_i}{dt} = \boldsymbol{F}_i \quad (2.29)$$

where

$$\boldsymbol{p}_i = \gamma_i m_i \boldsymbol{v}_i \quad \text{and} \quad \gamma_i = \frac{1}{\sqrt{1 - (v_i/c)^2}} \quad (2.30)$$

and the force $\boldsymbol{F}$ is typically dominated by the Lorentz force:

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \quad (2.31)$$

Poynting's theorem

The EM field and particles can do work on each other.

The flow of energy due to this work is given by Poynting's theorem,

$$\frac{\partial}{\partial t}\left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi}\right) = \nabla \cdot \boldsymbol{S} - \boldsymbol{j} \cdot \boldsymbol{E} \quad (2.32)$$

or in words:

The change of the energy density $(E^2 + B^2)/8\pi$ at a certain position equals the work $dW = \boldsymbol{j} \cdot \boldsymbol{E}\, dt$ done on matter by the EM field at that position plus the divergence of the Poynting vector.

where the Poynting vector, $\boldsymbol{S}$, is defined by

$$\boldsymbol{S} = \frac{c}{4\pi}(\boldsymbol{E} \times \boldsymbol{B}) \quad (2.33)$$

The units of S are erg cm^{-2} s^{-1}, i.e., power transported through an area, i.e., $S = dW/dA$), and $(E^2 + B^2)/8\pi$ is an energy density, i.e., has units erg cm^{-3}.

In order to derive Poynting's theorem, let's look at the mechanical work done on a particle i by the electric and the magnetic fields. This work is given by

$$\boldsymbol{v}_i\cdot\boldsymbol{F}_i = q_i\boldsymbol{v}_i\cdot\left(\boldsymbol{E}+\frac{\boldsymbol{v}_i}{c}\times\boldsymbol{B}\right) = q_i\boldsymbol{v}_i\cdot\boldsymbol{E} \quad (2.34)$$

since $\boldsymbol{v}_i\perp(\boldsymbol{v}_i\times\boldsymbol{B})$.
Summing over all particles at a certain position then gives

$$P=\sum_i\delta(\boldsymbol{x}-\boldsymbol{x}_i(t))q_i\boldsymbol{v}_i\cdot\boldsymbol{E}=\boldsymbol{j}\cdot\boldsymbol{E} \quad (2.35)$$

Because of Ampère's law,

$$\nabla\times\boldsymbol{B}=\frac{4\pi}{c}\boldsymbol{j}+\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} \quad\Rightarrow\quad \boldsymbol{j}=\frac{c}{4\pi}(\nabla\times\boldsymbol{B})-\frac{1}{4\pi}\frac{\partial\boldsymbol{E}}{\partial t} \quad (2.4)$$

we find

$$\boldsymbol{j}\cdot\boldsymbol{E}=\frac{c}{4\pi}\left(\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B})-\frac{1}{c}\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\right) \quad (2.36)$$

$$=\frac{c}{4\pi}\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B})-\frac{1}{8\pi}\frac{\partial(E^2)}{\partial t} \quad (2.37)$$

Analogously to $(a\times b)\cdot c=(b\times c)\cdot a$ one has

$$\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B})=\nabla_B\cdot(\boldsymbol{B}\times\boldsymbol{E})=-\nabla_B\cdot(\boldsymbol{E}\times\boldsymbol{B}) \quad (2.38)$$

where ∇_B operates only on $\boldsymbol{B}$. Therefore

$$\boldsymbol{j}\cdot\boldsymbol{E}=-\frac{c}{4\pi}\nabla_B\cdot(\boldsymbol{E}\times\boldsymbol{B})-\frac{1}{8\pi}\frac{\partial E^2}{\partial t} \quad (2.39)$$

Note that

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})=\nabla_B\cdot(\boldsymbol{E}\times\boldsymbol{B})+\nabla_E\cdot(\boldsymbol{E}\times\boldsymbol{B}) \quad (2.40)$$

Therefore, add $(c/4\pi)\,\nabla_E(\boldsymbol{E}\times\boldsymbol{B})$ to Eq. 2.39:

$$\boldsymbol{j}\cdot\boldsymbol{E}=-\frac{c}{4\pi}\Big(\nabla_B\cdot(\boldsymbol{E}\times\boldsymbol{B})+\nabla_E(\boldsymbol{E}\times\boldsymbol{B})-\nabla_E(\boldsymbol{E}\times\boldsymbol{B})\Big)-\frac{1}{8\pi}\frac{\partial E^2}{\partial t} \quad (2.41)$$

$$=-\frac{c}{4\pi}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})+\frac{c}{4\pi}\nabla_E(\boldsymbol{E}\times\boldsymbol{B})-\frac{1}{8\pi}\frac{\partial E^2}{\partial t} \quad (2.42)$$

but $\nabla_E(\boldsymbol{E}\times\boldsymbol{B})=(\nabla\times\boldsymbol{E})\cdot\boldsymbol{B}$ (similar to Eq. 2.38)

$$=-\frac{c}{4\pi}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})+\frac{c}{4\pi}(\nabla\times\boldsymbol{E})\cdot\boldsymbol{B}-\frac{1}{8\pi}\frac{\partial E^2}{\partial t} \quad (2.43)$$

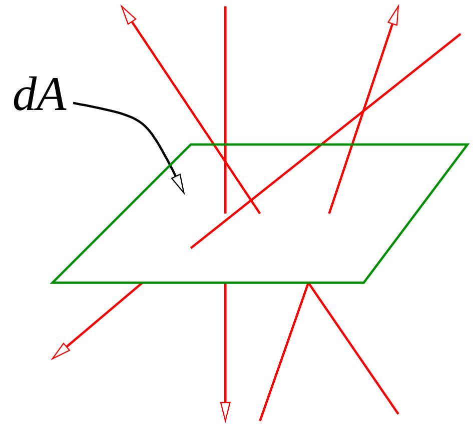
Calculate $\nabla\times\boldsymbol{E}$ using Faraday's law (Eq. 2.2), i.e.,

$$\frac{c}{4\pi}(\nabla\times\boldsymbol{E})\cdot\boldsymbol{B}=-\frac{1}{4\pi}\frac{\partial\boldsymbol{B}}{\partial t}\cdot\boldsymbol{B}=-\frac{1}{8\pi}\frac{\partial}{\partial t}(\boldsymbol{B}\cdot\boldsymbol{B})=-\frac{1}{8\pi}\frac{\partial B^2}{\partial t} \quad (2.44)$$

such that we finally obtain Poynting's theorem

$$-\boldsymbol{j}\cdot\boldsymbol{E}=\frac{c}{4\pi}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})+\frac{\partial}{\partial t}\left(\frac{E^2}{8\pi}+\frac{B^2}{8\pi}\right) \quad (2.32)$$

Flux Density, I



We now relate the electrodynamic quantities such as $\boldsymbol{E}$, $\boldsymbol{B}$, or $\boldsymbol{S}$ to measurables. Before we can do this, need to introduce some definitions.

Definition. Energy flux, F, is defined as the energy $\mathrm{d}E$ passing through area $\mathrm{d}A$ in time interval $\mathrm{d}t$:

$$\mathrm{d}E = F\,\mathrm{d}A\,\mathrm{d}t \quad (2.45)$$

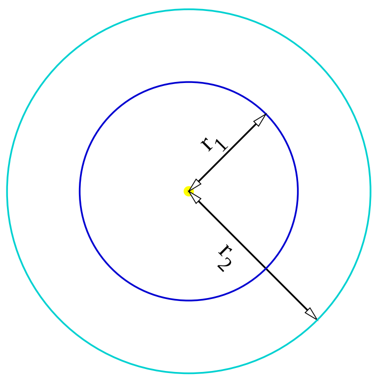
Units of F are erg cm^{-2} s^{-1}.

F depends on the *orientation of* $\mathrm{d}A$, and can also depend on the frequency.

Flux Density, II

Flux from an isotropic radiation source, i.e., a source emitting equal amounts of energy in all directions.

Spherically symmetric stars are isotropic radiation sources, other astronomical objects such as, e.g., Active Galactic Nuclei, are not.



Because of energy conservation, flux through two shells around source is identical:

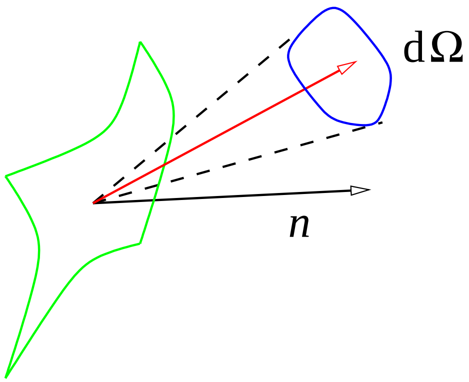
$$4\pi r_1^2\,F(r_1)=4\pi r_2^2 F(r_2) \quad (2.46)$$

and therefore we obtain the inverse square law,

$$F(r)=\frac{\text{const.}}{r^2} \quad (2.47)$$

Specific Intensity, I

Better description of radiation: energy carried along individual ray.



Problem: Rays are infinitely thin $\Longrightarrow$ No energy carried by them...

$\Longrightarrow$ Look at energy passing through area $\mathrm{d}A$ (with normal $\boldsymbol{n}$) in all rays going into spatial direction $\mathrm{d}\Omega$.

The specific intensity, I_ν, in the band $\nu, \ldots, \nu + \mathrm{d}\nu$ is defined via

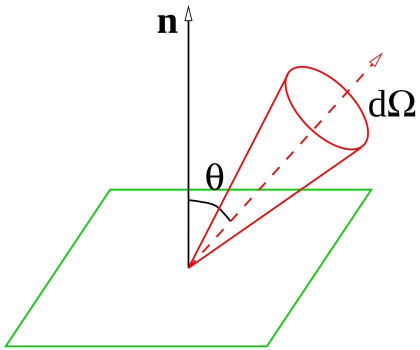
$$\mathrm{d}E = I_\nu \,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu \qquad (2.48)$$

I_ν is measured in units of erg $\mathrm{s}^{-1}\,\mathrm{cm}^{-2}\,\mathrm{sr}^{-1}\,\mathrm{Hz}^{-1}$ and depends on location, direction, and frequency.

In an isotropic radiation field, $I_\nu =$ const. for all directions.

Specific Intensity, II

Using the definition of I, we can calculate the net flux in a direction $\boldsymbol{n}$.

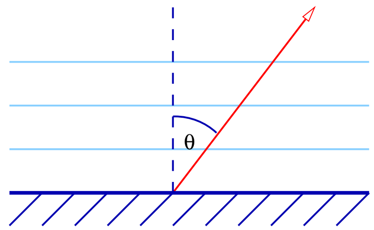


Contribution to flux in direction $\boldsymbol{n}$ from flux into direction $\mathrm{d}\Omega$:

$$\mathrm{d}F_\nu = I_\nu \cos\theta\,\mathrm{d}\Omega \qquad (2.49)$$

Integrate over all angles to obtain the total flux:

$$F_\nu = \int_{4\pi\,\mathrm{sr}} I_\nu \cos\theta\,\mathrm{d}\Omega = \int_{\theta=0}^{\pi}\int_0^{2\pi} I_\nu(\theta,\phi)\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \qquad (2.50)$$



Especially in the theory of stellar atmospheres, where often deals with the radiative transport through a slab of material and can assume cylindrical symmetry, one writes

$$\mu = \cos\theta \qquad (2.51)$$

where θ is the angle between the z-direction ("up"-"down" direction) and the direction of the light ray. Then

$$\frac{\mathrm{d}\mu}{\mathrm{d}\theta} = -\sin\theta \qquad (2.52)$$

and

$$F_\nu = \int_{\mu=+1}^{-1}\int_{\phi=0}^{2\pi} I_\nu(\mu,\theta)(-\mu)\sin\theta\frac{\mathrm{d}\mu}{\sin\theta}\,\mathrm{d}\phi \qquad (2.53)$$

$$= \int_{\mu=-1}^{+1}\int_{\phi=0}^{2\pi} I_\mu(\mu,\theta)\mu\,\mathrm{d}\mu\,\mathrm{d}\phi \qquad (2.54)$$

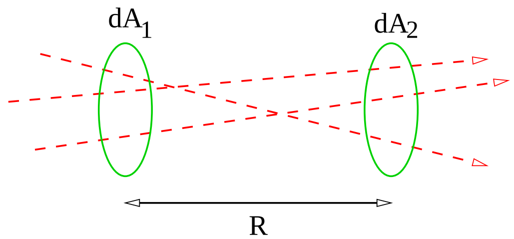
taking into account the cylindrical symmetry,

$$= 2\pi\int_{-1}^{+1} I_\nu(\mu)\mu\,\mathrm{d}\mu \qquad (2.55)$$

Specific Intensity, III

The specific intensity is constant along the line of sight.

Proof: Consider the following figure:



Energy carried through area $\mathrm{d}A_1$ *and* $\mathrm{d}A_2$ is given by

$$\mathrm{d}E_1 = I_1\,\mathrm{d}A_1\,\mathrm{d}t\,\mathrm{d}\Omega_1\,\mathrm{d}\nu \qquad (2.56)$$

$$\mathrm{d}E_2 = I_2\,\mathrm{d}A_2\,\mathrm{d}t\,\mathrm{d}\Omega_2\,\mathrm{d}\nu \qquad (2.57)$$

where $\mathrm{d}\Omega_1$: solid angle subtended by $\mathrm{d}A_2$ at $\mathrm{d}A_1$, and vice versa:

$$\mathrm{d}\Omega_1 = \mathrm{d}A_2/R^2 \qquad (2.58)$$

$$\mathrm{d}\Omega_2 = \mathrm{d}A_1/R^2 \qquad (2.59)$$

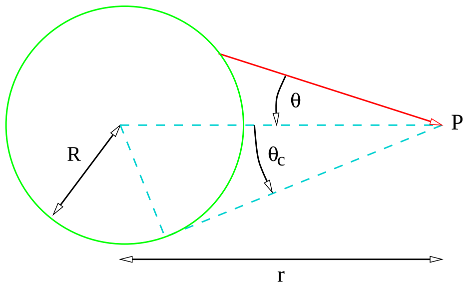
Energy conservation implies $\mathrm{d}E_1 = \mathrm{d}E_2$, i.e.,

$$I_1\,\mathrm{d}A_1\,\mathrm{d}t\frac{\mathrm{d}A_2}{R^2}\,\mathrm{d}\nu = I_2\,\mathrm{d}A_2\,\mathrm{d}t\,\frac{\mathrm{d}A_1}{R^2}\,\mathrm{d}\nu \qquad (2.60)$$

that is: $I_1 = I_2 =$ const.. QED.

Specific Intensity, IV

$I =$ const. along a ray does not contradict the inverse square law!



Proof: Assume sphere of uniform brightness, $\mathscr{B}$, the flux measured at point P is the flux from all visible points of sphere:

$$F = \int I \cos\theta \, d\Omega \quad (2.61)$$

$$= \mathscr{B} \cdot \int_0^{\pi} \int_0^{\theta_c} \sin\theta \cos\theta \, d\theta \, d\phi \quad (2.62)$$

where $\sin\theta_c = R/r$. Therefore

$$F = 2\pi\mathscr{B} \int_0^{\arcsin(R/r)} \sin\theta \cos\theta \, d\theta \quad (2.63)$$

but because $\int_0^{\alpha} \sin x \cos x \, dx = \frac{1}{2}\sin^2\alpha$,

$$= 2\pi\mathscr{B} \cdot \frac{1}{2}\left(\frac{R}{r}\right)^2 = \pi\mathscr{B}\left(\frac{R}{r}\right)^2 \propto \frac{1}{r^2} \quad (2.64)$$

$\Longrightarrow$ inverse square law is consequence of decreasing solid angle of objects!

2–13

One consequence of Eq. 2.64 is that the flux on the surface of a source of uniform brightness is given by

$$F = \pi\mathscr{B} \quad (2.65)$$

Therefore, especially for stellar atmospheres, one sometimes defines the astrophysical flux,

$$\mathscr{F} := \frac{F}{\pi} \quad (2.66)$$

such that for stars (which roughly have uniform surface brightness),

$$\mathscr{F} = \mathscr{B} \quad (2.67)$$

Be aware of this source of possible confusion!

Energy Density and Mean Intensity

The last important radiation quantity (for the moment) is the energy density, u_ν. For a certain direction, Ω, and volume element dV, u_ν is defined via

$$dE = u_\nu(\Omega) \, dV \, d\Omega \, d\nu \quad (2.68)$$

But for light, the volume element can be written as

$$dV = c \, dt \cdot dA \quad (2.69)$$

such that Eq. 2.68 becomes

$$dE = c u_\nu(\Omega) \, d\Omega \, dt \, d\nu \quad (2.70)$$

Compare this to the definition of the intensity:

$$dE = I_\nu \, dA \, dt \, d\Omega \, d\nu \quad (2.48)$$

Therefore,

$$u_\nu(\Omega) = I_\nu / c \quad (2.71)$$

Energy Density and Mean Intensity

The total energy density at frequency ν is then

$$u_\nu = \int_{4\pi\,\mathrm{sr}} u_\nu(\Omega) \, d\Omega = \frac{1}{c} \int_{4\pi\,\mathrm{sr}} I_\nu(\Omega) \, d\Omega =: \frac{4\pi}{c} J_\nu \quad (2.72)$$

where the mean intensity is defined by

$$J_\nu = \frac{1}{4\pi} \int_{4\pi\,\mathrm{sr}} I_\nu(\Omega) \, d\Omega \quad (2.73)$$

Note that for an isotropic radiation field, $I_\nu(\Omega) = I_\nu$, $I_\nu = J_\nu$.

The total radiation density is obtained by integration:

$$u = \int u_\nu \, d\nu = \frac{4\pi}{c} \int_0^\infty J_\nu \, d\nu \quad (2.74)$$

Introduction

After defining radiation quantities, we can now study the transport of radiation ("radiative transfer").

Three effects influence radiation

1. emission of radiation
2. absorption of radiation
3. scattering

Scattering can be treated as combination of emission and absorption, will not talk about that here.

Emission

Radiation can be emitted, adding energy to the beam:

$$\mathrm{d}E_\nu = j_\nu \,\mathrm{d}V \,\mathrm{d}\Omega \,\mathrm{d}t \tag{2.75}$$

where j_ν is coefficient for spontaneous emission, i.e., energy added per unit volume, unit angle, and unit time, units of j_ν are erg cm^{-3} s^{-1} sr^{-1} Hz^{-1}.

The change in intensity is

$$\mathrm{d}I_\nu = j_\nu \,\mathrm{d}s \tag{2.76}$$

(writing $\mathrm{d}V = \mathrm{d}A \,\mathrm{d}s$)

Note that j_ν *depends on direction!*

For an isotropic emitter only (e.g., randomly oriented sources):

$$j_\nu = \frac{1}{4\pi} P_\nu \tag{2.77}$$

where P_ν is the radiated power per unit volume.

Absorption

Radiation is also absorbed.

Medium with particle number density n (cm^{-3}), each having effective absorbing area (cross section) σ_ν (cm^2):

- Number of absorbers: $n \,\mathrm{d}A \,\mathrm{d}s$
- Total absorbing area: $n\sigma_\nu \,\mathrm{d}A \,\mathrm{d}s$

$\Longrightarrow$ Energy absorbed out of beam:

$$-\,\mathrm{d}I_\nu \,\mathrm{d}A \,\mathrm{d}\Omega \,\mathrm{d}t \,\mathrm{d}\nu = I n\sigma_\nu \,\mathrm{d}A \,\mathrm{d}s \,\mathrm{d}\Omega \,\mathrm{d}t \,\mathrm{d}\nu \tag{2.78}$$

or

$$\mathrm{d}I_\nu = -n\sigma_\nu I_\nu \,\mathrm{d}s =: -\alpha_\nu I_\nu \,\mathrm{d}s \tag{2.79}$$

where α_ν: absorption coefficient (units cm^{-1}).

2–18

In the study of stellar interiors, one often encounters the angle integrated emissivity, ϵ_ν, defined as the emitted energy *per unit mass*,

$$\mathrm{d}E_\nu = \epsilon_\nu \rho \,\mathrm{d}V \,\mathrm{d}t \,\mathrm{d}t \frac{\mathrm{d}\Omega}{4\pi} \tag{2.80}$$

where the factor $\mathrm{d}\Omega/4\pi$ is the fraction of energy emitted into direction $\mathrm{d}\Omega$.

For isotropic emission,

$$\epsilon_\nu = \frac{\rho}{4\pi j_\nu} \tag{2.81}$$

Likewise, the absorptivity is also measured per unit mass of absorbing material, through

$$\alpha_\nu = n\rho \frac{\sigma_\nu}{\rho} = \rho \cdot \frac{n\sigma_\nu}{\rho} =: \rho\kappa_\nu \tag{2.82}$$

where κ_ν is the mass absorption coefficient, or the opacity coefficient, and has units of cm^2 g^{-1}.

Equation of radiative transfer

Combining Eqs. 2.76 and 2.79 gives

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu \qquad (2.83)$$

the equation of radiative transfer (RT)

For pure emission, $\alpha_\nu = 0$, such that

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu \qquad (2.84)$$

Separation of variables and assuming the path starts at $s = 0$ gives:

$$I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')\,\mathrm{d}s' \qquad (2.85)$$

For pure emission, the brightness increase is equal to the integrated emissivity along the line of sight.

Optical Depth, I

For pure absorption, Eq. (2.83) is

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu(s) \qquad (2.86)$$

At depth s of a medium irradiated with $I(s=0) = I_0$, separation of variables gives

$$I_\nu(s) = I_0 \exp\left(-\int_0^s \alpha_\nu(s')\,\mathrm{d}s'\right) =: I_0 \mathrm{e}^{-\tau_\nu} \qquad (2.87)$$

For pure absorption, the intensity decreases exponentially with the optical depth.

The optical depth, τ, is defined by

$$\tau_\nu(s) = \int_0^s \alpha_\nu(s')\,\mathrm{d}s' = \int_0^s n(s')\sigma_\nu\,\mathrm{d}s' = n\sigma_\nu s \qquad (2.88)$$

where the last step is valid for a homogeneous medium.

τ is central to describing most effects in RT.

$\tau > 1 \Longrightarrow$ "medium is optically thick or "opaque"
$\tau < 1 \Longrightarrow$ "medium is optically thin or "transparent"

Optical Depth, II

Exponential absorption law $\Longrightarrow$ Probability for photon to travel distance $> \tau$: $\exp(-\tau)$.

$\Longrightarrow$ Mean optical depth traveled:

$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu \exp(-\tau_\nu)\,\mathrm{d}\tau_\nu = 1 \qquad (2.89)$$

The path length corresponding to $\tau = 1$ is called the mean free path,

$$\tau_\nu = n\sigma_\nu l_\nu \quad \Longrightarrow \quad \langle l\rangle = \frac{1}{n\sigma_\nu} \qquad (2.90)$$

In an inhomogeneous medium, the mean free path is defined locally.

Example: In the center of the Sun, $\rho \sim 150\,\mathrm{g\,cm^{-3}}$, (wrongly) assuming pure H, this means $n \sim 9 \times 10^{25}$ particles $\mathrm{cm^{-3}}$. The cross section is on the order of $10^{-25}\,\mathrm{cm}^2$ (see later) $\Longrightarrow$ $\langle l\rangle_\odot \sim 1$ mm. Compare this to a solar radius of 7×10^{10} cm...

Formal Solution

It is useful to write RT in terms of τ.

$\tau_\nu = \int_0^s \alpha_\nu(s')\,\mathrm{d}s'$ implies $\mathrm{d}s = \mathrm{d}\tau_\nu/\alpha_\nu$, such that

$$\alpha_\nu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = j_\nu - \alpha_\nu I_\nu \qquad (2.91)$$

and therefore,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu =: S_\nu - I_\nu \qquad (2.92)$$

where S_ν is called the source function.

One can obtain a formal solution of the transfer equation in terms of S (see manuscript):

$$I_\nu(\tau_\nu) = I_\nu(0)\mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} \mathrm{e}^{-(\tau_\nu-\tau'_\nu)} S(\tau'_\nu)\,\mathrm{d}\tau'_\nu \qquad (2.93)$$

Interpretation:
The intensity emerging from an absorbing medium equals the irradiated intensity, corrected for absorption *plus* the (absorbed) integrated source contributions.

Since α_ν can contain effects from stimulated emission ($\propto I_\nu$, see later), the formal solution can not be obtained for any meaningful physical system.

The derivation of Eq. 2.93 is straightforward algebra:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = S_\nu - I_\nu \tag{2.94}$$

$$\mathrm{e}^{\tau_\nu}\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = \mathrm{e}^{\tau_\nu}S_\nu - \mathrm{e}^{\tau_\nu}I_\nu \tag{2.95}$$

$$\mathrm{e}^{\tau_\nu}\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} + \mathrm{e}^{\tau_\nu}I_\nu = \mathrm{e}^{\tau_\nu}S_\nu \tag{2.96}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\nu}\left(\mathrm{e}^{\tau_\nu}I_\nu\right) = \mathrm{e}^{\tau_\nu}S_\nu \tag{2.97}$$

Separation of variables gives

$$\mathrm{e}^{\tau_\nu}I_\nu(\tau_\nu) - I_\nu(0) = \int_0^{\tau_\nu} \mathrm{e}^{\tau'_\nu}S_\nu(\tau'_\nu)\,\mathrm{d}\tau'_\nu \tag{2.98}$$

such that finally

$$I_\nu(\tau_\nu) = I_\nu(0)\mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} \mathrm{e}^{-(\tau_\nu-\tau'_\nu)}S(\tau'_\nu)\,\mathrm{d}\tau'_\nu \tag{2.93}$$

Formal Solution

If the radiation is in complete thermodynamical equilibrium, nothing is allowed to change, i.e., Eq. 2.92 reads

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = S_\nu - I_\nu = 0 \tag{2.99}$$

such that

In thermodynamic equilibrium: $S_\nu = I_\nu = B_\nu$

where B_ν is the Planck function,

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1} \tag{3.31}$$

to be derived next.

If the assumption $S_\nu = B_\nu$ is made locally, one speaks of local thermodynamic equilibrium, LTE.