World Models

Structure

Observations: cosmological principle holds: The universe is homogeneous and isotropic.

$\Longrightarrow$ Need theoretical framework obeying the cosmological principle.

Use combination of

- General Relativity
- Thermodynamics
- Quantum Mechanics

$\Longrightarrow$ Complicated!

For 99% of the work, the above points can be dealt with separately:

1. Define metric obeying cosmological principle.
2. Obtain equation for evolution of universe using Einstein field equations.
3. Use thermo/QM to obtain equation of state.
4. Solve equations.

GRT vs. Newton

Before we can start to think about universe: Brief introduction to assumptions of general relativity.

$\Longrightarrow$ See theory lectures for the gory details, or check with the literature (Weinberg or MTW).

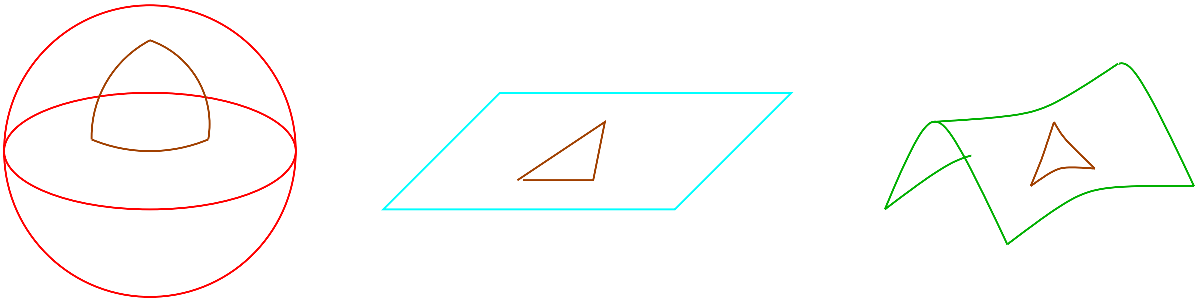
Assumptions of GRT:

- Space is 4-dimensional, might be curved
- Matter (=Energy) modifies space (Einstein field equation).
- Covariance: physical laws must be formulated in a coordinate-system independent way.
- Strong equivalence principle: There is no experiment by which one can distinguish between free falling coordinate systems and inertial systems.
- At each point, space is locally Minkowski (i.e., locally, SRT holds).

$\Longrightarrow$ Understanding of geometry of space necessary to understand physics.

2D Metrics

Before describing the 4D geometry of the universe: first look at two-dimensional spaces (easier to visualize).



After Silk (1997, p. 107)

There are three classes of isotropic and homogeneous two-dimensional spaces:

- 2-sphere ($\mathscr{S}^2$) positively curved
- x-y-plane ($\mathbb{R}^2$) zero curvature
- hyperbolic plane ($\mathscr{H}^2$) negatively curved

(curvature $\approx \sum$angles in triangle $>$, $=$, or $<$ 180°)

We will now compute what the metric for these spaces looks like.

2D Metrics

The metric describes the local geometry of a space.

Differential distance, ds, in Euclidean space, $\mathbb{R}^2$:

$$ds^2 = dx_1^2 + dx_2^2 \quad (4.1)$$

The metric tensor, $g_{\mu\nu}$, is defined via

$$ds^2 = \sum_\mu \sum_\nu g_{\mu\nu}\, dx^\mu\, dx^\nu =: g_{\mu\nu}\, dx^\mu\, dx^\nu \quad (4.2)$$

(Einstein's summation convention)

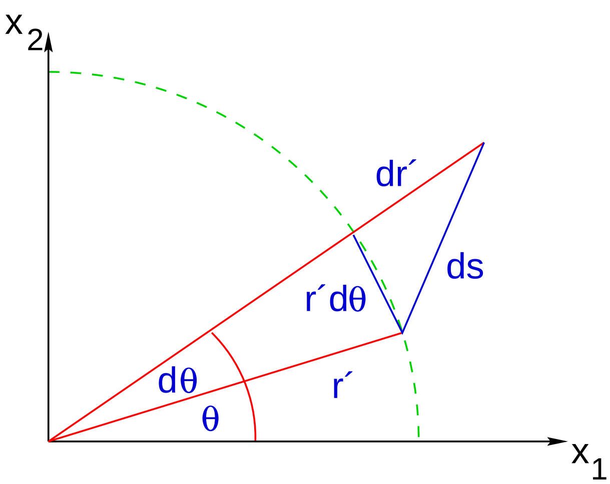
Thus, for the $\mathbb{R}^2$,

$$\begin{aligned} g_{11} &= 1 \qquad & g_{12} &= 0 \\ g_{21} &= 0 \qquad & g_{22} &= 1 \end{aligned} \quad (4.3)$$

But: Other coordinate-systems possible!

Changing to polar coordinates r', θ, defined by

$$x_1 =: r' \cos\theta \quad \text{and} \quad x_2 =: r' \sin\theta \quad (4.4)$$



it is easy to see that

$$ds^2 = dr'^2 + r'^2\, d\theta^2 \quad (4.5)$$

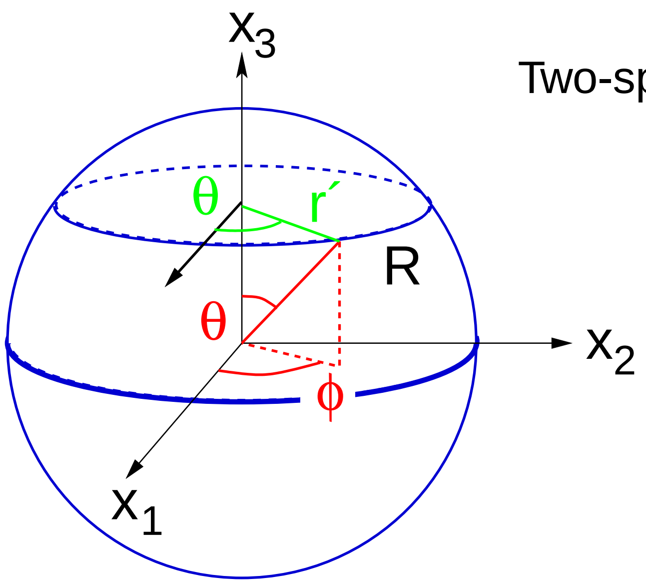
substituting $r' = Rr$,

(change of scale)

$$ds^2 = R\{dr^2 + r^2\, d\theta^2\} \quad (4.6)$$

2D Metrics

A more complicated case occurs if space is curved. *Easiest case:* surface of three-dimensional sphere (a two-sphere).



After Kolb & Turner (1990, Fig. 2.1)

Two-sphere with radius R in $\mathbb{R}^3$:

$$x_1^2 + x_2^2 + x_3^2 = R^2 \quad (4.7)$$

Length element of $\mathbb{R}^3$:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

Eq. (4.7) gives

$$x_3 = \sqrt{R^2 - x_1^2 - x_2^2}$$

such that

$$dx_3 = \frac{\partial x_3}{\partial x_1}\, dx_1 + \frac{\partial x_3}{\partial x_2}\, dx_2 = -\frac{x_1\, dx_1 + x_2\, dx_2}{\sqrt{R^2 - x_1^2 - x_2^2}} \quad (4.8)$$

Introduce again polar coordinates r', θ in x_3-plane:

$$x_1 =: r' \cos\theta \quad \text{and} \quad x_2 =: r' \sin\theta \quad (4.4)$$

(note: r', θ only unique in upper or lower half-sphere)

The differentials are given by

$$\begin{aligned} dx_1 &= \cos\theta\, dr' - r' \sin\theta\, d\theta \\ dx_2 &= \sin\theta\, dr' + r' \cos\theta\, d\theta \end{aligned} \quad (4.9)$$

2D Metrics

In cartesian coordinates, the length element on $\mathscr{S}^2$ is

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \frac{(x_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_2)^2}{R^2 - x_1^2 - x_2^2} \tag{4.10}$$

inserting eq. (4.9) gives after some algebra

$$= r'^2\,\mathrm{d}\theta^2 + \frac{R^2}{R^2 - r'^2}\,\mathrm{d}r'^2 \tag{4.11}$$

finally, defining $r = r'/R$ (i.e., $0 \le r \le 1$) results in

$$\mathrm{d}s^2 = R^2 \left\{ \frac{\mathrm{d}r^2}{1 - r^2} + r^2\,\mathrm{d}\theta^2 \right\} \tag{4.12}$$

Alternatively, we can work in spherical coordinates on $\mathscr{S}^2$

$$\begin{aligned} x_1 &= R\sin\theta\cos\phi \\ x_2 &= R\sin\theta\sin\phi \\ x_3 &= R\cos\theta \end{aligned} \tag{4.13}$$

($\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$).
Going through the same steps as before, we obtain after some tedious algebra

$$\mathrm{d}s^2 = R^2 \left\{ \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \right\} \tag{4.14}$$

2D Metrics

(Important) remarks:

1. The 2-sphere has no edges, has no boundaries, but has still a finite volume, $V = 4\pi R^2$.
2. Expansion or contraction of sphere caused by variation of $R \Longrightarrow R$ determines the *scale* of volumes and distances on $\mathscr{S}^2$.

   R is called the *scale factor*

3. Positions on $\mathscr{S}^2$ are defined, e.g., by r and θ, *independent* on the value of R

   r and θ are called *comoving coordinates*

4. Although the metrics Eq. (4.10), (4.12), and (4.14) look *very* different, they still describe the *same* space $\Longrightarrow$ that's why physics should be covariant.

2D Metrics

The hyperbolic plane, $\mathscr{H}^2$, is defined by

$$x_1^2 + x_2^2 - x_3^2 = -R^2 \tag{4.15}$$

If we work in Minkowski space, where

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2 \tag{4.16}$$

then

$$= \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \frac{(x_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_2)^2}{R^2 + x_1^2 + x_2^2} \tag{4.17}$$

$\Longrightarrow$ substitute $R \to iR$ (where $i = \sqrt{-1}$) to obtain same form as for sphere (eq. 4.11)!

Therefore,

$$\mathrm{d}s^2 = R^2 \left\{ \frac{\mathrm{d}r^2}{1 + r^2} + r^2\,\mathrm{d}\theta^2 \right\} \tag{4.18}$$

2D Metrics

The analogy to spherical coordinates on the hyperbolic plane are given by

$$\begin{aligned} x_1 &= R \sinh\theta \cos\phi \\ x_2 &= R \sinh\theta \sin\phi \\ x_3 &= R \cosh\theta \end{aligned} \tag{4.19}$$

($\theta \in [-\infty, +\infty]$, $\phi \in [0, 2\pi]$).

A session with Maple (see handout) will convince you that these coordinates give

$$\mathrm{d}s^2 = R^2 \left\{ \mathrm{d}\theta^2 + \sinh^2\theta \, \mathrm{d}\phi^2 \right\} \tag{4.20}$$

Remark:
$\mathscr{H}^2$ is unbound and has an infinite volume.

Transcript of Maple session to obtain Eq. (4.20):

```
> x1:=r*sinh(theta)*cos(phi);
```

$$x1 := r\sinh(\theta)\cos(\phi)$$

```
> x2:=r*sinh(theta)*sin(phi);
```

$$x2 := r\sinh(\theta)\sin(\phi)$$

```
> x3:=r*cosh(theta);
```

$$x3 := r\cosh(\theta)$$

```
> dx1:=diff(x1,theta)*dtheta+diff(x1,phi)*dphi;
```

$$dx1 := r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi$$

```
> dx2:=diff(x2,theta)*dtheta+diff(x2,phi)*dphi;
```

$$dx2 := r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi$$

```
> ds2:=dx1*dx1+dx2*dx2-(x1*dx1+x2*dx2)^2/(r^2+x 1^2+x2^2);
```

$$\begin{aligned}ds2 := {} & (r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi)^2 \\ & + (r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi)^2 - (\\ & r\sinh(\theta)\cos(\phi)\,(r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi) \\ & + r\sinh(\theta)\sin(\phi)\,(r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi))^2 \Big/ (\\ & r^2 + r^2\sinh(\theta)^2\cos(\phi)^2 + r^2\sinh(\theta)^2\sin(\phi)^2)\end{aligned}$$

```
> expand(ds2);
```

$$\begin{aligned} & r^2\cosh(\theta)^2\cos(\phi)^2\,dtheta^2 + r^2\sinh(\theta)^2\sin(\phi)^2\,dphi^2 + r^2\cosh(\theta)^2\sin(\phi)^2\,dtheta^2 \\ & + r^2\sinh(\theta)^2\cos(\phi)^2\,dphi^2 - \frac{r^4\sinh(\theta)^2\cos(\phi)^4\cosh(\theta)^2\,dtheta^2}{\%1} \\ & - 2\,\frac{r^4\sinh(\theta)^2\cos(\phi)^2\cosh(\theta)^2\,dtheta^2\sin(\phi)^2}{\%1} - \frac{r^4\sinh(\theta)^2\sin(\phi)^4\cosh(\theta)^2\,dtheta^2}{\%1} \\ & \%1 := r^2 + r^2\sinh(\theta)^2\cos(\phi)^2 + r^2\sinh(\theta)^2\sin(\phi)^2\end{aligned}$$

```
> simplify(",{cosh(theta)^2-sinh(theta)^2=1}, [cosh(theta)]);
```

$$r^2\,dtheta^2 + r^2\sinh(\theta)^2\,dphi^2$$

2D Metrics

To summarize:

Sphere:

$$ds^2 = R^2 \left\{ \frac{dr^2}{1 - r^2} + r^2\, d\theta^2 \right\} \qquad (4.12)$$

Plane:

$$ds^2 = R^2 \left\{ dr^2 + r^2\, d\theta^2 \right\} \qquad (4.6)$$

Hyperbolic Plane:

$$ds^2 = R^2 \left\{ \frac{dr^2}{1 + r^2} + r^2\, d\theta^2 \right\} \qquad (4.18)$$

$\Longrightarrow$ All three metrics can be written as

$$ds^2 = R^2 \left\{ \frac{dr^2}{1 - k\, r^2} + r^2\, d\theta^2 \right\} \qquad (4.21)$$

where k defines the geometry:

$$k = \begin{cases} +1 & \text{spherical} \\ 0 & \text{planar} \\ -1 & \text{hyperbolic} \end{cases} \qquad (4.22)$$

2D Metrics

For "spherical coordinates" we found:

Sphere:

$$\mathrm{d}s^2 = R^2 \left\{ \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right\} \tag{4.14}$$

Plane:

$$\mathrm{d}s^2 = R^2 \left\{ \mathrm{d}\theta^2 + \theta^2 \mathrm{d}\phi^2 \right\} \tag{4.6}$$

Hyperbolic plane:

$$\mathrm{d}s^2 = R^2 \left\{ \mathrm{d}\theta^2 + \sinh^2\theta \, \mathrm{d}\phi^2 \right\} \tag{4.20}$$

$\Longrightarrow$ All three metrics can be written as

$$\mathrm{d}s^2 = R^2 \left\{ \mathrm{d}\theta^2 + S_k^2(\theta) \, \mathrm{d}\phi^2 \right\} \tag{4.23}$$

where

$$S_k(\theta) = \begin{cases} \sin\theta & \text{for } k = +1 \\ \theta & \text{for } k = 0 \\ \sinh\theta & \text{for } k = -1 \end{cases} \tag{4.24}$$

We will also need the $\cos$-like analogue

$$C_k(\theta) = \sqrt{1 - kS_k^2(\theta)} = \begin{cases} \cos\theta & \text{for } k = +1 \\ 1 & \text{for } k = 0 \\ \cosh\theta & \text{for } k = -1 \end{cases} \tag{4.25}$$

Note that, compared to the earlier formulae, some coordinates have been renamed. This is confusing, but legal...

RW Metric

- Cosmological principle $+$ expansion $\Longrightarrow$ $\exists$ freely expanding cosmical coordinate system.
  - Observers $=:$ fundamental observers
  - Time $=:$ cosmic time

  This is the coordinate system in which the 3K radiation is isotropic, clocks can be synchronized, e.g., by adjusting time to the local density of the universe.

  $\Longrightarrow$ Metric has temporal and spatial part.

  This also follows directly from the equivalence principle.

- *Homogeneity and isotropy* $\Longrightarrow$ spatial part is spherically symmetric:

$$\mathrm{d}\psi^2 := \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \qquad (4.26)$$

- *Expansion:* $\exists$ scale factor, $R(t)$ $\Longrightarrow$ measure distances using comoving coordinates.

$\Longrightarrow$ metric looks like

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R^2(t)\left[f^2(r)\,\mathrm{d}r^2 + g^2(r)\,\mathrm{d}\psi^2\right] \qquad (4.27)$$

where $f(r)$ and $g(r)$ are arbitrary.

UWarwick

RW Metric

Metrics of the form of eq. (4.27) are called Robertson-Walker (RW) metrics (1935).

Previously studied by Friedmann and Lemaître. . .

One common choice is

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R^2(t)\left[\mathrm{d}r^2 + S_k^2(r)\,\mathrm{d}\psi^2\right] \quad (4.28)$$

where

$R(t)$**:** scale factor, containing the physics

t**:** cosmic time

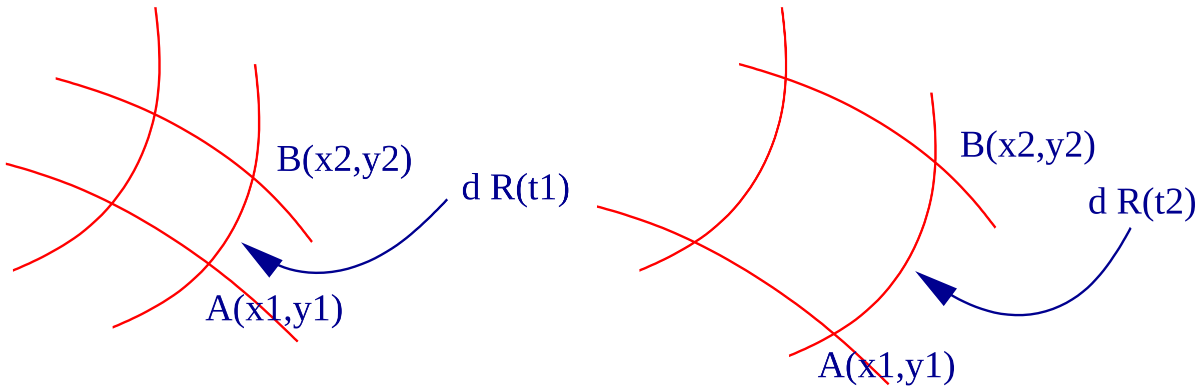
r**,** θ**,** ϕ**:** comoving coordinates

$S_k(r)$ was defined in Eq. (4.24).

Remark: θ and ϕ describe directions on sky, as seen from the arbitrary center of the coordinate system (=us), r can be interpreted as a radial coordinate.

RW Metric

The RW metric defines an universal coordinate system tied to expansion of space:



Scale factor $R(t)$ describes evolution of universe.

- d is called the comoving distance.
- $D(t) := d \cdot R(t)$ is called the proper distance,

(note that R is unitless, i.e., d and $dR(t)$ are measured in Mpc)

"World model": $R(t)$ from GRT *plus* assumptions about physics.

RW Metric

Other forms of the RW metric are also used:

1. Substitution $S_k(r) \longrightarrow r$ gives

$$ds^2 = c^2\, dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\, d\psi^2\right\} \tag{4.29}$$

(i.e., other definition of comoving radius r).

2. A metric with a dimensionless scale factor,

$$a(t) := \frac{R(t)}{R(t_0)} = \frac{R(t)}{R_0} \tag{4.30}$$

(where t_0=today, i.e., $a(t_0) = 1$), gives

$$ds^2 = c^2\, dt^2 - a^2(t)\left\{dr^2 + \frac{S_k^2(R_0 r)}{R_0^2}\, d\psi^2\right\} \tag{4.31}$$

3. Using $a(t)$ *and* the substitution $S_k(r) \longrightarrow r$ is also possible:

$$ds^2 = c^2\, dt^2 - a^2(t)\left\{\frac{dr^2}{1-k\cdot(R_0 r)^2} + r^2\, d\psi^2\right\} \tag{4.32}$$

The units of $R_0 r$ are Mpc $\Longrightarrow$ *Used for observations!*

UWarwick

RW Metric

4. Replace cosmic time, t, by conformal time,
   $\mathrm{d}\eta = \mathrm{d}t/R(t) \Longrightarrow$ conformal metric,

$$\mathrm{d}s^2 = R^2(\eta)\left\{\mathrm{d}\eta^2 - \frac{\mathrm{d}r^2}{1-kr} - r^2\,\mathrm{d}\psi^2\right\} \tag{4.33}$$

   Theoretical importance of this metric: For $k = 0$, i.e., a flat space, the RW metric $=$ Minkowski line element $\times$ $R^2(\eta) \Longrightarrow$ Equivalence principle!

5. Finally, the metric can also be written in the isotropic form,

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \frac{R(t)}{1+(k/4)r^2}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\psi^2\right\} \tag{4.34}$$

   Here, the term in $\{\ldots\}$ is just the line element of a 3d-sphere $\Longrightarrow$ isotropy!

Note: There are as many notations as authors, e.g., some use $a(t)$ where we use $R(t)$, etc. $\Longrightarrow$ Be careful!

Note 2: Local homogeneity and isotropy (i.e., within a Hubble radius, $r = c/H_0$), do not imply *global* homogeneity and isotropy $\Longrightarrow$ Cosmologies with a non-trivial topology are possible (e.g., also with more dimensions...).

Hubble's Law

Hubble's Law follows from the variation of $R(t)$:

Small scales $\Longrightarrow$ Euclidean geometry

Proper distance between two observers:

$$D(t) = d \cdot R(t) \tag{4.35}$$

where d: comoving distance.

Expansion $\Longrightarrow$ proper separation changes:

$$\frac{\Delta D}{\Delta t} = \frac{R(t+\Delta t)d - R(t)d}{\Delta t} \tag{4.36}$$

Thus, for $\Delta t \rightarrow 0$,

$$v = \frac{\mathrm{d}D}{\mathrm{d}t} = \dot{R}\,d = \frac{\dot{R}}{R}\,D =: H\,D \tag{4.37}$$

$\Longrightarrow$ Identify local Hubble "constant" as

$$H = \frac{\dot{R}}{R} = \dot{a}(t) \tag{4.38}$$

($a(t)$ from Eq. 4.30, $a(\text{today}) = 1$)

Since $R = R(t) \Longrightarrow H$ is time-dependent!

For small v, interpreted classically the red-shift is

$$z = 1 + \frac{v}{c} \quad \Longrightarrow \quad z - 1 = \frac{Hd}{c} \tag{4.39}$$

UWarwick

Redshift, I

The cosmological redshift is a consequence of the expansion of the universe:

The comoving distance is constant, thus in terms of the proper distance:

$$d = \frac{D(t = \text{today})}{R(t = \text{today})} = \frac{D(t)}{R(t)} = \text{const.} \quad (4.40)$$

Set $a(t) = R(t)/R(t = \text{today})$, then eq. (4.40) implies

$$\lambda_{\text{obs}} = \frac{\lambda_{\text{emit}}}{a_{\text{emit}}} \quad (4.41)$$

(λ_{obs}: observed wavelength, λ_{emit}: emitted wavelength)

Thus the observed redshift is

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} - 1 \quad (4.42)$$

or

$$1 + z = \frac{1}{a_{\text{emit}}} = \frac{R(t = \text{today})}{R(t)} \quad (4.43)$$

Light emitted at $z = 1$ was emitted when the universe was half as big as today!

z: measure for *relative size* of universe at time the observed light was emitted.

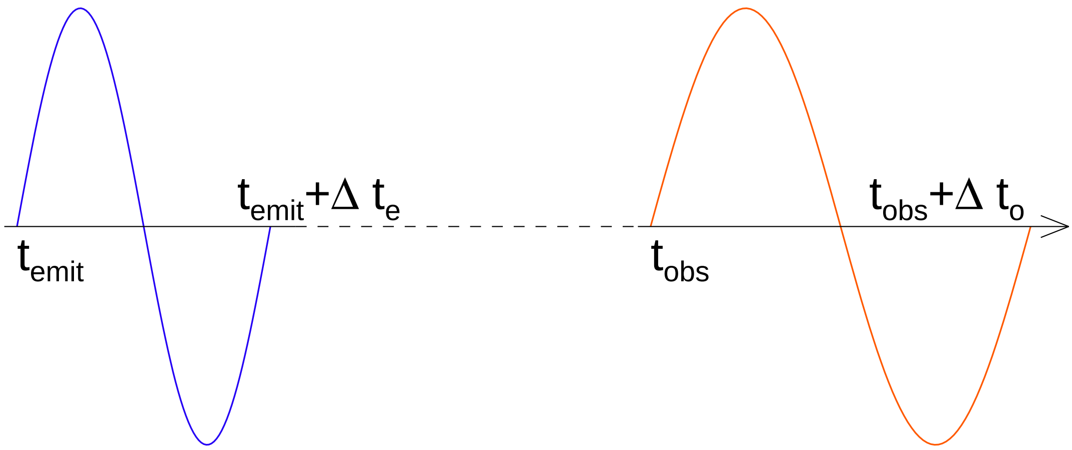
Because of $z = \nu_{\text{emit}}/\nu_{\text{obs}}$,

$$\frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} = \frac{1}{a_{\text{emit}}} \quad (4.44)$$

An alternative derivation of the cosmological redshift follows directly from general relativity, using the basic GR fact that for photons d$s^2 = 0$. Inserting this into the metric, and assuming without loss of generality that d$\psi^2 = 0$, one finds

$$0 = c^2 \,\mathrm{d}t^2 - R^2(t)\,\mathrm{d}r^2 \quad\Longrightarrow\quad \mathrm{d}r = \pm\frac{c\,\mathrm{d}t}{R(t)} \tag{4.45}$$

Since photons travel forward, we choose the $+$-sign.



The *comoving* distance traveled by photons emitted at cosmic times t_emit and $t_\mathrm{emit} + \Delta t_\mathrm{e}$ is

$$r_1 = \int_{t_\mathrm{emit}}^{t_\mathrm{obs}} \frac{c\,\mathrm{d}t}{R(t)} \quad\text{and}\quad r_2 = \int_{t_\mathrm{emit}+\Delta t_\mathrm{e}}^{t_\mathrm{obs}+\Delta t_\mathrm{o}} \frac{c\,\mathrm{d}t}{R(t)} \tag{4.46}$$

But the comoving distances are equal, $r_1 = r_2$! Therefore

$$0 = \int_{t_\mathrm{emit}}^{t_\mathrm{obs}} \frac{c\,\mathrm{d}t}{R(t)} - \int_{t_\mathrm{emit}+\Delta t_\mathrm{e}}^{t_\mathrm{obs}+\Delta t_\mathrm{o}} \frac{c\,\mathrm{d}t}{R(t)} \tag{4.47}$$

$$= \int_{t_\mathrm{emit}}^{t_\mathrm{emit}+\Delta t_\mathrm{e}} \frac{c\,\mathrm{d}t}{R(t)} - \int_{t_\mathrm{obs}}^{t_\mathrm{obs}+\Delta t_\mathrm{o}} \frac{c\,\mathrm{d}t}{R(t)} \tag{4.48}$$

If Δt small $\Longrightarrow R(t) \approx$ const.:

$$= \frac{c\,\Delta t_\mathrm{e}}{R(t_\mathrm{emit})} - \frac{c\,\Delta t_\mathrm{o}}{R(t_\mathrm{obs})} \tag{4.49}$$

For a wave: $c\Delta t = \lambda$, such that

$$\frac{\lambda_\mathrm{emit}}{R(t_\mathrm{emit})} = \frac{\lambda_\mathrm{obs}}{R(t_\mathrm{obs})} \quad\Longleftrightarrow\quad \frac{\lambda_\mathrm{emit}}{\lambda_\mathrm{obs}} = \frac{R(t_\mathrm{emit})}{R(t_\mathrm{obs})} \tag{4.50}$$

From this equation it is straightforward to derive Eq. (4.42).

Redshift, II

Outside of the local universe: Eq. (4.43) only valid interpretation of z.

$\Longrightarrow$ It is common to interpret z as in special relativity:

$$1+z=\sqrt{\frac{1+v/c}{1-v/c}} \tag{4.51}$$

This is WRONG

Redshift is due to expansion of space, not due to motion of galaxy.

What *is* true is that z is accumulation of many infinitesimal red-shifts à la Eq. (4.39), see, e.g., Peacock (1999).

Time Dilatation

Note the implication of Eq. (4.49) on the hand-out:

$$\frac{c\,\Delta t_{\rm e}}{R(t_{\rm emit})} = \frac{c\,\Delta t_{\rm o}}{R(t_{\rm obs})} \tag{4.49}$$

$\Longrightarrow$ dt/R is constant:

$$\frac{\mathrm{d}t}{R(t)} = \text{const.} \tag{4.52}$$

In other words:

$$\frac{\mathrm{d}t_{\rm obs}}{\mathrm{d}t_{\rm emit}} = \frac{R(t_{\rm obs})}{R(t_{\rm emit})} = 1 + z \tag{4.53}$$

$\Longrightarrow$ Time dilatation of events at large z.

This cosmological time dilatation has been observed in the light curves of supernova outbursts.

All other observables apart from z (e.g., number density $N(z)$, luminosity distance $d_{\rm L}$, etc.) require explicit knowledge of $R(t)$ $\Longrightarrow$ Need to look at the dynamics of the universe.

Friedmann Equations, I

General relativistic approach: Insert metric into Einstein equation to obtain differential equation for $R(t)$:

Einstein equation:

$$\underbrace{R_{\mu\nu} - \frac{1}{2}\mathscr{R} g_{\mu\nu}}_{G_{\mu\nu}} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu} \qquad (4.54)$$

where

$g_{\mu\nu}$: Metric tensor ($\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$)

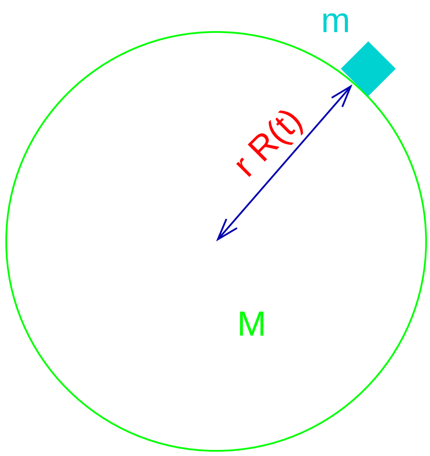
- $R_{\mu\nu}$: Ricci tensor (function of $g_{\mu\nu}$)
- $\mathscr{R}$: Ricci scalar (function of $g_{\mu\nu}$)
- $G_{\mu\nu}$: Einstein tensor (function of $g_{\mu\nu}$)

$T_{\mu\nu}$: Stress-energy tensor, describing curvature of space due to fields present (matter, radiation,...)

Λ: Cosmological constant

$\Longrightarrow$ Messy, but doable

Friedmann Equations, II



Here, Newtonian derivation of Friedmann equations: Dynamics of a mass element on the surface of sphere of density $\rho(t)$ and comoving radius d, i.e., proper radius $d \cdot R(t)$ (after McCrea & Milne, 1934).

Mass of sphere:

$$M = \frac{4\pi}{3}(d\,R)^3\rho(t) = \frac{4\pi}{3}d^3\rho_0 \quad \text{where} \quad \rho(t) = \frac{\rho_0}{R(t)^3} \tag{4.55}$$

Force on mass element:

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(d\,R(t)) = -\frac{GMm}{(dR(t))^2} = -\frac{4\pi G}{3}\frac{d\rho_0}{R^2(t)}m \tag{4.56}$$

Canceling $m \cdot d$ gives momentum equation:

$$\ddot{R} = -\frac{4\pi G}{3}\frac{\rho_0}{R^2} = -\frac{4\pi G}{3}\rho(t)R(t) \tag{4.57}$$

From energy conservation, or from multiplying Eq. (4.57) with $\dot{R}$ and integrating, we obtain the energy equation,

$$\begin{aligned}\frac{1}{2}\dot{R}^2 &= +\frac{4\pi G}{3}\frac{\rho_0}{R(t)} + \text{const.}\\ &= +\frac{4\pi G}{3}\rho(t)R^2(t) + \text{const.}\end{aligned} \tag{4.58}$$

where the constant can only be obtained from GR.

Friedmann Equations, III

Problems with the Newtonian derivation:

1. Cloud is implicitly assumed to have $r_{\text{cloud}} < \infty$ (for $r_{\text{cloud}} \rightarrow \infty$ the force is undefined)
   $\Longrightarrow$ violates cosmological principle.
2. Particles move *through* space
   $\Longrightarrow v > c$ possible
   $\Longrightarrow$ violates SRT.

Why do we get correct result?

GRT $\longrightarrow$ Newton for small scales and mass densities; since universe is isotropic $\Longrightarrow$ scale invariance on Mpc scales $\Longrightarrow$ Newton sufficient (classical limit of GR).

(In fact, point 1 above *does* hold in GR: Birkhoff's theorem).

Friedmann Equations, IV

The exact GR derivation of Friedmanns equation gives:

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \left[\frac{1}{3}\Lambda R\right]$$
$$\dot{R}^2 = +\frac{8\pi G\rho}{3}R^2 - kc^2 + \left[\frac{1}{3}\Lambda c^2 R^2\right] \qquad (4.59)$$

Notes:

1. For $k = 0$: Eq. (4.59) $\longrightarrow$ Eq. (4.58).
2. $k \in \{-1, 0, +1\}$ determines the curvature of space.
3. The density, ρ, includes the contribution of all different kinds of energy (remember mass-energy equivalence!).
4. There is energy associated with the vacuum, parameterized by the parameter Λ.

The evolution of the Hubble parameter is ($\Lambda = 0$):

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2(t) = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} \qquad (4.60)$$

The Critical Density, I

Solving Eq. (4.60) for k:

$$\frac{R^2}{c}\left(\frac{8\pi G}{3}\rho - H^2\right) = k \tag{4.61}$$

$\Longrightarrow$ Sign of curvature parameter k only depends on density, ρ:

Defining

$$\rho_\mathrm{c} = \frac{3H^2}{8\pi G} \quad \text{and} \quad \Omega = \frac{\rho}{\rho_\mathrm{c}} \tag{4.62}$$

it is easy to see that:

$$\begin{aligned} \Omega > 1 &\Longrightarrow k > 0 \quad \text{closed} \\ \Omega = 1 &\Longrightarrow k = 0 \quad \text{flat} \\ \Omega < 1 &\Longrightarrow k < 0 \quad \text{open} \end{aligned}$$

thus ρ_c is called the critical density.

For $\Omega \leq 1$ the universe will expand until ∞,
for $\Omega > 1$ we will see the "big crunch".

Current value of ρ_c: $\sim 1.67 \times 10^{-24}\,\mathrm{g/cm^3}$,
(3...10 H-atoms/m^3).
Measured: $\Omega = 0.1 \ldots 0.3$.

(but note that Λ can influence things ($\Omega_\Lambda = 0.7$)!).

The Critical Density, II

Ω has a second order effect on the expansion:

Taylor series of $R(t)$ around $t = t_0$:

$$\frac{R(t)}{R(t_0)} = \frac{R(t_0)}{R(t_0)} + \frac{\dot{R}(t_0)}{R(t_0)}(t - t_0) + \frac{1}{2}\frac{\ddot{R}(t_0)}{R(t_0)}(t - t_0)^2 \qquad (4.63)$$

The Friedmann equation Eq. (4.57) can be written

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho = -\frac{4\pi G}{3}\Omega\frac{3H^2}{8\pi G} = -\frac{\Omega H^2}{2} \qquad (4.64)$$

Since $H(t) = \dot{R}/R$ (Eq. 4.38), Eq. (4.63) is

$$\frac{R(t)}{R(t_0)} = 1 + H_0\,(t - t_0) - \frac{1}{2}\frac{\Omega_0}{2}H_0^2\,(t - t_0)^2 \qquad (4.65)$$

where $H_0 = H(t_0)$ and $\Omega_0 = \Omega(t_0)$.

The subscript 0 is often omitted in the case of Ω.

Often, Eq. (4.65) is written using the deceleration parameter:

$$q := \frac{\Omega}{2} = -\frac{\ddot{R}(t_0)R(t_0)}{\dot{R}^2(t_0)} \qquad (4.66)$$

UWarwick

Equation of state, I

For the evolution of the universe, need to look at three different kinds of equation of state:

Matter: Normal particles get diluted by expansion of the universe:

$$\rho_{\mathrm{m}} \propto R^{-3} \qquad (4.67)$$

Matter is also often called dust by cosmologists.

Radiation: The energy density of radiation decreases because of volume expansion and because of the cosmological redshift (Eq. 4.50: $\lambda_{\mathrm{o}}/\lambda_{\mathrm{e}} = \nu_{\mathrm{e}}/\nu_{\mathrm{o}} = R(t_{\mathrm{o}})/R(t_{\mathrm{e}})$) $\Longrightarrow$

$$\rho_{\mathrm{r}} \propto R^{-4} \qquad (4.68)$$

Vacuum: The vacuum energy density (=Λ) is independent of R:

$$\rho_{\mathrm{v}} = \text{const.} \qquad (4.69)$$

Inserting these equations of state into the Friedmann equation and solving with the boundary condition $R(t=0)=0$ then gives a specific world model.

Equation of state, II

Current scale factor is determined by H_0 and Ω_0:
Friedmann for $t = t_0$:

$$\dot{R}_0^2 - \frac{8\pi G}{3}\rho R_0^2 = -kc^2 \qquad (4.70)$$

Insert Ω and note $H_0 = \dot{R}_0/R_0$

$$\Longleftrightarrow H_0^2 R_0^2 - H_0^2 \Omega_0 R_0^2 = -kc^2 \qquad (4.71)$$

And therefore

$$R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega - 1}} \qquad (4.72)$$

For $\Omega \longrightarrow 0$, $R_0 \longrightarrow c/H_0$, the Hubble length.

For $\Omega = 1$, R_0 is arbitrary.

We now have everything we need to solve the Friedmann equation and determine the evolution of the universe. Three cases: $k = 0, +1, -1$.

$k = 0$, Matter dominated

For the matter dominated, flat case (the Einstein-de Sitter case), the Friedmann equation is

$$\dot{R}^2 - \frac{8\pi G}{3}\frac{\rho_0 R_0^3}{R^3} R^2 = 0 \tag{4.73}$$

For $k = 0$: $\Omega = 1$ and

$$\frac{8\pi G\rho_0}{3} = \Omega_0 H_0^2 R_0^3 = H_0^2 R_0^3 \tag{4.74}$$

Therefore, the Friedmann eq. is

$$\dot{R}^2 - \frac{H_0^2 R_0^3}{R} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}R}{\mathrm{d}t} = H_0 R_0^{3/2} R^{-1/2} \tag{4.75}$$

Separation of variables and setting $R(0) = 0$,

$$\int_0^{R(t)} R^{1/2}\,\mathrm{d}R = H_0 R_0^{3/2} t \quad \Longleftrightarrow \quad \frac{2}{3} R^{3/2}(t) = H_0 R_0^{3/2} t \tag{4.76}$$

Such that

$$R(t) = R_0 \left(\frac{3H_0}{2}\,t\right)^{2/3} \tag{4.77}$$

For $k = 0$, the universe expands until ∞, its current age ($R(t_0) = R_0$) is given by

$$t_0 = \frac{2}{3H_0} \tag{4.78}$$

Reminder: The Hubble-Time is $H_0^{-1} = 9.78\,\mathrm{Gyr}/h$.

$k = +1$, Matter dominated, I

For the matter dominated, closed case, Friedmanns equation is

$$\dot{R}^2 - \frac{8\pi G}{3}\frac{\rho_0 R_0^3}{R} = -c^2 \quad \Longleftrightarrow \quad \dot{R}^2 - \frac{H_0^2 R_0^3 \Omega_0}{R} = -c^2 \tag{4.79}$$

Inserting R_0 from Eq. (4.72) gives

$$\dot{R}^2 - \frac{H_0^2 c^3 \Omega_0}{H_0^3(\Omega - 1)^{3/2}}\frac{1}{R} = -c^2 \tag{4.80}$$

which is equivalent to

$$\frac{dR}{dt} = c\left(\frac{\xi}{R} - 1\right)^{1/2} \quad \text{with} \quad \xi = \frac{c}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{4.81}$$

With the boundary condition $R(0) = 0$, separation of variables gives

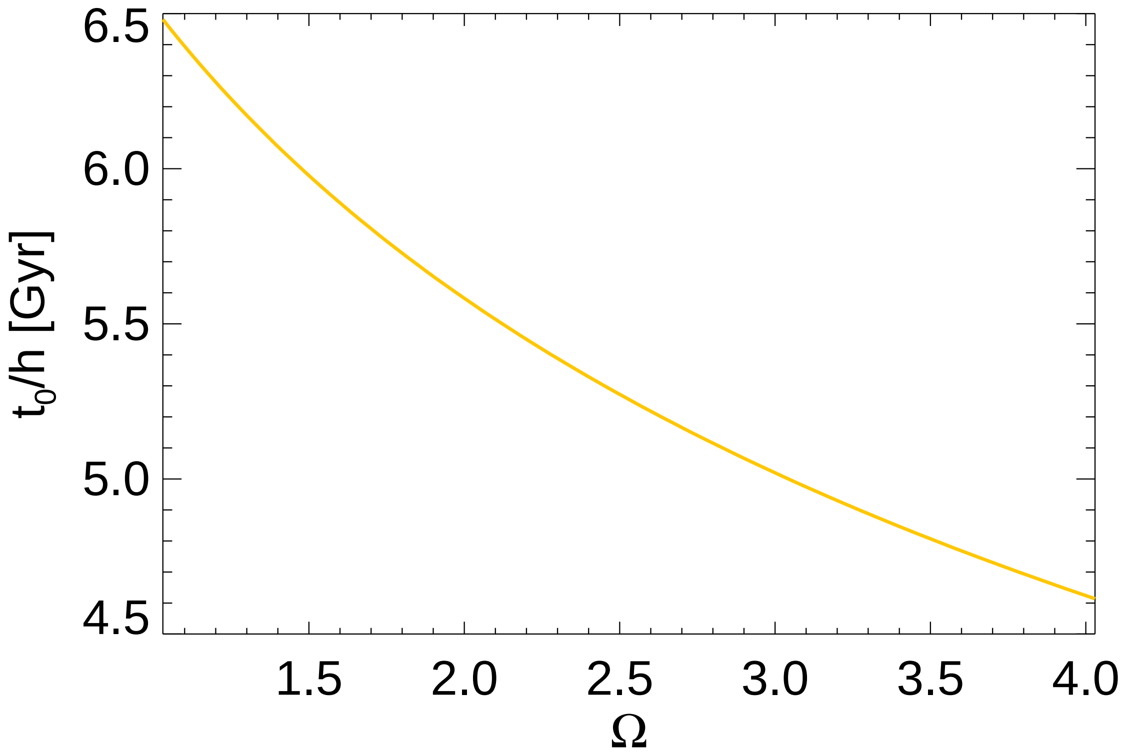
$$ct = \int_0^{R(t)} \frac{dR}{(\xi/R - 1)^{1/2}} = \int_0^{R(t)} \frac{\sqrt{R}\,dR}{(\xi - R)^{1/2}} \tag{4.82}$$

Integration by substitution gives

$$R = \xi \sin^2\frac{\theta}{2} = \frac{\xi}{2}(1 - \cos\theta)$$
$$\Longrightarrow \quad ct = \frac{\xi}{2}(\theta - \sin\theta) \tag{4.83}$$

UWarwick

$k = +1$, Matter dominated, II



The age of the universe, t_0, is obtained by solving

$$R_0 = \frac{c}{H_0(\Omega_0 - 1)^{1/2}}$$
$$= \frac{\xi}{2}(1 - \cos\theta_0) = \frac{1}{2}\frac{c}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(1 - \cos\theta_0) \quad (4.84)$$

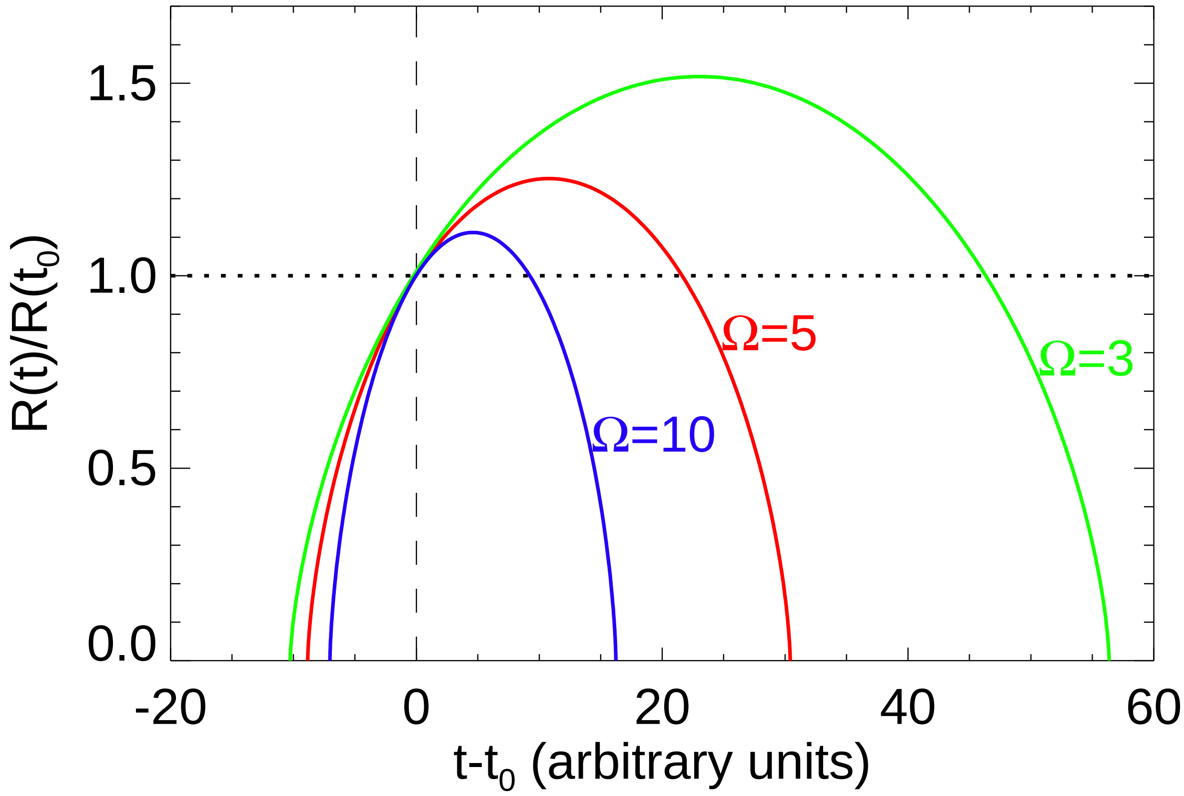
(remember Eq. 4.72!). Therefore

$$\cos\theta_0 = \frac{2 - \Omega_0}{\Omega_0} \iff \sin\theta_0 = \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1} \quad (4.85)$$

Inserting this into Eq. (4.83) gives

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\left[\arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) - \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1}\right] \quad (4.86)$$

$k = +1$, Matter dominated, III



Since R is a cyclic function $\Longrightarrow$ The closed universe has a finite lifetime.

Max. expansion at $\theta = \pi$, with a maximum scale factor of

$$R_{\max} = \xi = \frac{c}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{4.87}$$

After that: contraction to the big crunch at $\theta = 2\pi$.

$\Longrightarrow$ The lifetime of the closed universe is

$$t = \frac{\pi}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{4.88}$$

$k = -1$, Matter dominated, I

Finally, the matter dominated, open case. This case is very similar to the case of $k = +1$:

For $k = -1$, the Friedmann equation becomes

$$\frac{\mathrm{d}R}{\mathrm{d}t} = c\left(\frac{\zeta}{R} + 1\right)^{1/2} \tag{4.89}$$

where

$$\zeta = \frac{c}{H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}} \tag{4.90}$$

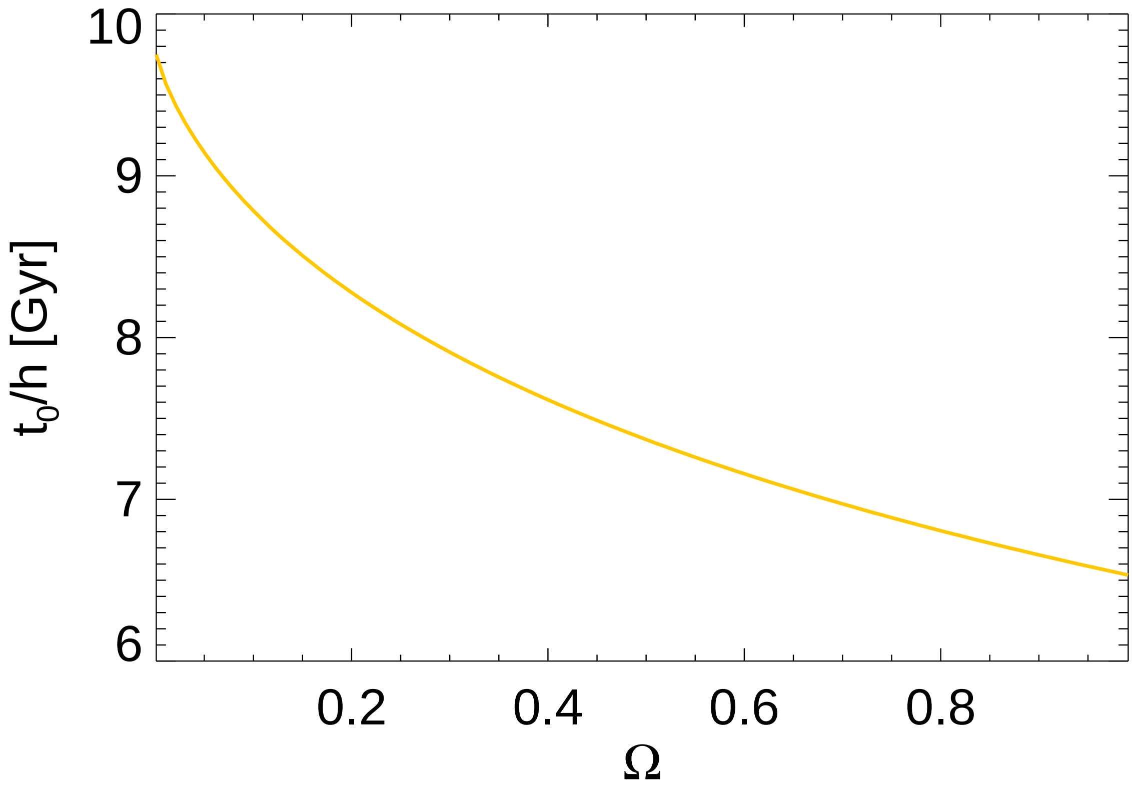
Separation of variables gives after a little bit of algebra

$$\begin{aligned} R &= \frac{\zeta}{2}(\cosh\theta - 1) \\ ct &= \frac{\zeta}{2}(\sinh\theta - 1) \end{aligned} \tag{4.91}$$

where the integration was again performed by substitution.

Note: θ here has *nothing* to do with the coordinate angle θ!

$k = -1$, Matter dominated, II



To obtain the age of the universe, note that at the present time,

$$\cosh\theta_0 = \frac{2-\Omega_0}{\Omega_0}$$
$$\sinh\theta_0 = \frac{2}{\Omega_0}\sqrt{1-\Omega_0} \tag{4.92}$$

(identical derivation as that leading to Eq. 4.84) such that

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}} \cdot \left\{\frac{2}{\Omega_0}\sqrt{1-\Omega_0} - \ln\left(\frac{2-\Omega_0+2\sqrt{1-\Omega_0}}{\Omega_0}\right)\right\} \tag{4.93}$$

UWarwick

Summary

For the matter dominated case, our results from Eqs. (4.83), and (4.91) can be written with the functions S_k and C_k (Eqs. 4.24 and 4.25):

$$\begin{aligned} R &= k\mathscr{R}\left(1 - C_k(\theta)\right) \\ ct &= k\mathscr{R}\left(\theta - S_k(\theta)\right) \end{aligned} \tag{4.94}$$

where

$$S_k(\theta) = \begin{cases} \sin\theta \\ \theta \\ \sinh\theta \end{cases} \quad \text{and} \quad C_k(\theta) = \begin{cases} \cos\theta & \text{for } k = +1 \\ 1 & \text{for } k = 0 \\ \cosh\theta & \text{for } k = -1 \end{cases} \tag{4.24, 4.25}$$

Eq. (4.94) is called the cycloid solution.

The characteristic radius, $\mathscr{R}$, is given by

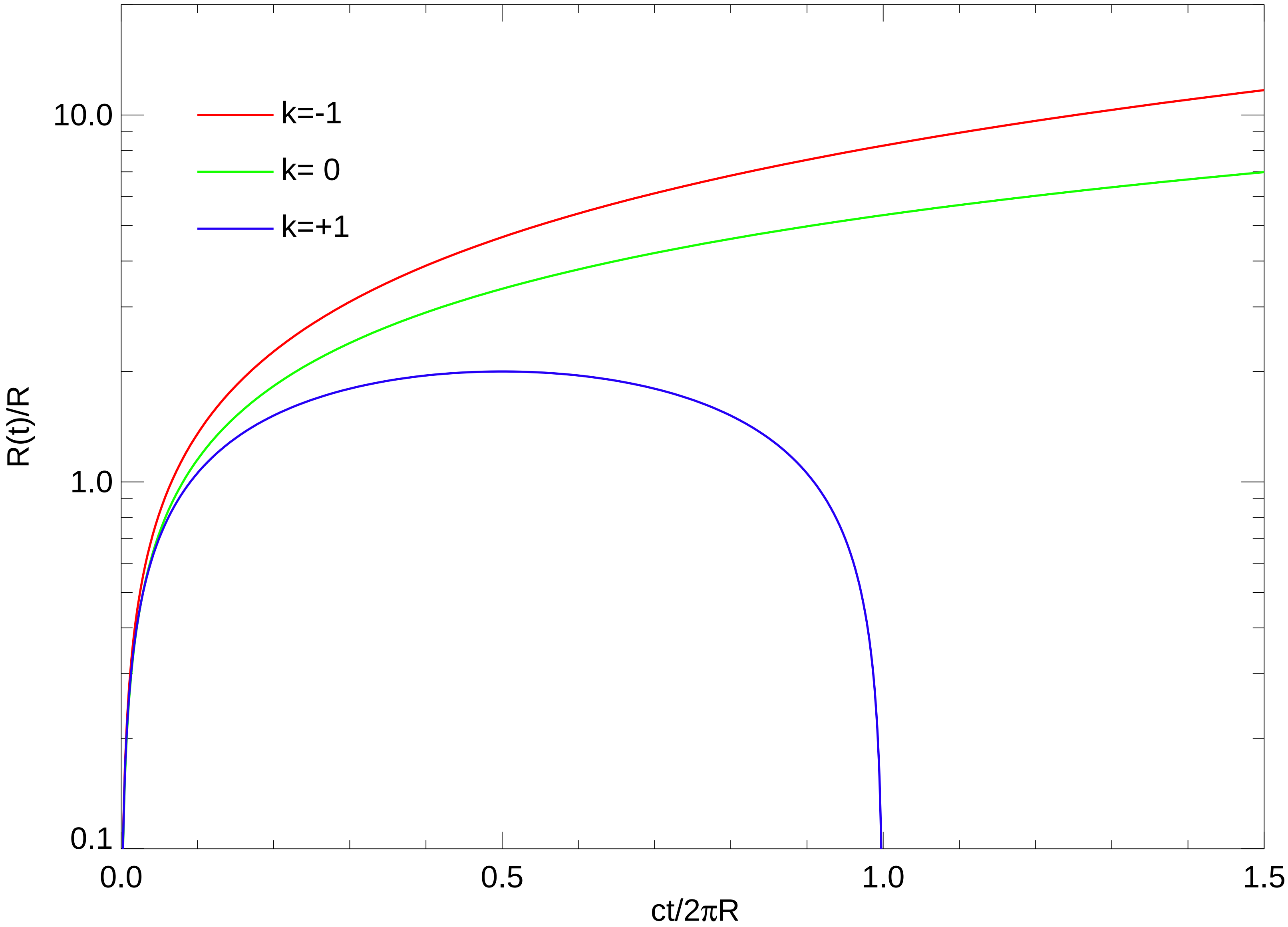
$$\mathscr{R} = \frac{c}{H_0} \frac{\Omega_0/2}{\left(k\left(\Omega_0 - 1\right)\right)^{3/2}} \tag{4.95}$$

(note typo in Eq. 3.42 of Peacock, 1999).

Notes:

1. Eq. (4.94) can also be derived as the result of the Newtonian collapse/expansion of a spherical mass distribution.
2. θ is called the development angle, it can be shown to be equal to the *conformal time* of Eq. (4.33).

Summary



Bibliography

McCrea, W. H., & Milne, E. A., 1934, Quart. J. Math. (Oxford Series), 5, 73

Silk, J., 1997, A Short History of the Universe, Scientific American Library 53, (New York: W. H. Freeman)